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A POSTERIORI ERROR ESTIMATES OF A NON-CONFORMING FINITE ELEMENT METHOD FOR PROBLEMS WITH ARTIFICIAL BOUNDARY CONDITIONS*

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Abstract

An a posteriori error estimator is obtained for a nonconforming finite element approximation of a linear elliptic problem, which is derived from a corresponding unbounded domain problem by applying a nonlocal approximate artificial boundary condition. Our method can be easily extended to obtain a class of a posteriori error estimators for various conforming and nonconforming finite element approximations of problems with different artificial boundary conditions. The reliability and efficiency of our a posteriori error estimator are rigorously proved and are verified by numerical examples.

Mathematics subject classification: 65N15, 65N30, 65N38.
Key words: a posteriori estimate, nonconforming finite element method, artificial boundary conditions

1. Introduction

Many physical and engineering problems such as the electric field and the magnetic field, can be modelled by PDEs on unbounded domains. To efficiently solve such problems by numerical methods, one often introduces proper artificial boundary conditions to translate these problems to bounded domain ones [9,10]. These artificial boundary conditions often have implicit integral forms, which are quite different from those of explicit boundary conditions: Dirichlet, Neumann, or mixed boundary conditions.

Furthermore, when the solutions of the reduced bounded domain problems have some singularities, e.g., singularities arising from re-entrant corners, and singularity of Green's function, adaptive mesh refinement strategy can be very useful to improve the efficiency of the finite element approximations. In this case, a posteriori estimators are often required to identify the regions which need further refinement. There are many different methods for the a posteriori estimation, e.g., the residual estimates [4, 12], the averaging methods [12, 14, 15], etc., however, they are mostly developed for bounded domain problems imposed with explicit boundary conditions.

In this paper, we will develop, for the first time to our knowledge, a reliable and efficient a posteriori estimator for a non-conforming finite element approximation of bounded domain elliptic problems with (at least part of) the boundary conditions given in an implicit integral form. Such problems come naturally from unbounded domain elliptic problems by imposing proper implicit artificial boundary conditions. For simplicity, we consider only a model exterior problem in two dimensions. Our approach, however, also easily applies to more general problems

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defined on unbounded domains, such as problems of the potential of the stray field energy in micromagnetics [11, 13] and the semi-strip field of stationary flow in a channel [10], etc..

The rest of the paper is organized as follows. In Section 2, we illustrate how to apply an artificial boundary method to a unbounded domain model problem to produce an equivalent bounded domain problem with an implicit artificial boundary condition [9–11]. In Section 3, inspired by [4], we introduce an equivalent mixed problem, which serves as a useful tool for the a posteriori error estimation. In Section 4, a non-conforming finite element method for the reduced bounded domain problem is briefly discussed. In Section 5, an a posteriori error estimator for the non-conforming finite element approximation of the model problem is given, and its reliability and efficiency are proved. In Section 6, some numerical examples are given to verify our theoretical results.

2. The Model Problem and the Artificial Boundary Method

We consider a general second-order linear elliptic problem [10]

$$-\mathrm{div}(A\nabla u) + cu = f, \quad \text{in } \Omega, \tag{2.1}$$

$$(A\nabla u)\cdot \mathbf{n} = g, \quad \text{on } \Gamma_N, \tag{2.2}$$

$$u = u_D, \quad \text{on } \Gamma_D, \tag{2.3}$$

$$u - u_\infty \to 0, \quad \text{as } \|x\| \to \infty, \tag{2.4}$$

where $\Omega \subset \mathbf{R}^2$ is a unbounded domain with a Lipschitz boundary $\Gamma = \Gamma_D \bigcup \Gamma_N$ satisfying that the Dirichlet boundary Γ_D is closed, $\Gamma_D \bigcap \Gamma_N = \emptyset$ and the length of the Dirichlet boundary $|\Gamma_D| > 0$ whenever $\Gamma_D \neq \emptyset$, $\mathbf{n}$ is the unit exterior normal to Γ_N, u_∞ is in general a unknown constant and $u_\infty = 0$ when $\Gamma_D = \emptyset$, $c \in L^\infty(\Omega)$ is non-negative and satisfies $c(x) \geq c_0 \geq 0$ for almost all $x \in \Omega$, and the coefficient matrix $A \in L^\infty(\Omega; \mathbf{R}^{2\times 2})$ is symmetric and uniformly positive definite, that is, for some constants $0 < \mu < M < \infty$, there holds

$$\mu\|y\|^2 \leq y \cdot A(x)y \leq M\|y\|^2, \qquad \forall y \in \mathbf{R}^2 \text{ and for a.e. } x \in \Omega.$$

Furthermore, we assume that $\mathrm{supp}(f)$, $\mathrm{supp}(A-I)$, and $\mathrm{supp}(c-c_0)$ are compact, where $\mathrm{supp}(\cdot)$ denotes the support set of a given function, and I is the identity matrix.

For such a problem, if R is sufficiently large so that $\mathrm{supp}(f) \cup \mathrm{supp}(A-I) \cup \mathrm{supp}(c-c_0) \cup \Gamma \subset B(0,R) := \{x : \|x\| < R\}$, then the circle $\Gamma_e := \{x : \|x\| = R\}$ can be taken as an artificial boundary, which divides the unbounded domain Ω into two parts $\Omega_i := \Omega \bigcap B(0,R)$ and $\Omega_e = \{x : \|x\| > R\}$. Artificial boundary conditions can be introduced on $\Gamma_e = \partial B(0,R)$. For simplicity and without loss of generality, we restrict ourselves to the case when $c_0 = 0$, similar artificial boundary conditions for the general case can be found in [10].

Since the solution u to problem (2.1)-(2.4) is harmonic in Ω_e, we have, for $r > R$,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{2.5}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\cos n\phi \, d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\sin n\phi \, d\phi, \tag{2.6}$$

are the Fourier coefficients of $u(R,\theta)$, and we have $u_\infty = a_0/2$.

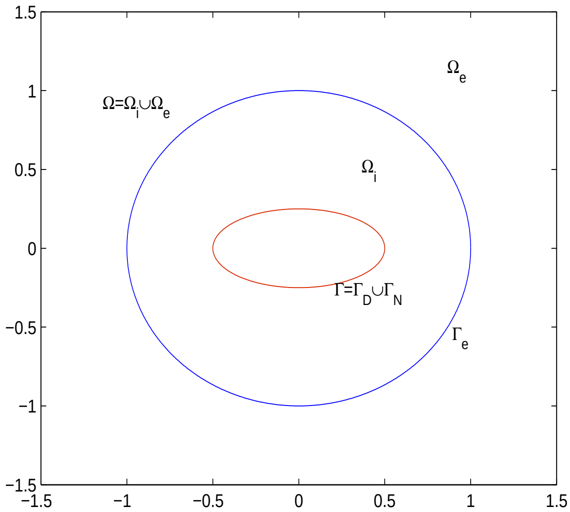

Fig. 2.1. The artificial boundary

Differentiating (2.5) with respect to r and setting $r = R$, we obtain

$$\frac{\partial u(R,\theta)}{\partial r} = -\frac{1}{\pi R}\sum_{n=1}^{\infty} n \int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)d\phi \equiv \mathcal{B}u(R,\theta), \tag{2.7}$$

where $\mathcal{B} : H^{1/2}(\Gamma_e) \to H^{-1/2}(\Gamma_e)$ is a bounded operator [7]. Let $\mathbf{n}$ be the unit exterior normal to Γ_e with respect to Ω_i. Then by (2.7) and the fact that u is harmonic in a neighborhood of Γ_e, we obtain an artificial boundary condition

$$\frac{\partial u(R,\phi)}{\partial \mathbf{n}} = \mathcal{B}u(R,\phi). \tag{2.8}$$

Thus, the unbounded domain problem (2.1)-(2.4) is reduced to the following equivalent problem defined on the bounded domain Ω_i

$$-\mathrm{div}(A\nabla u) + cu = f, \quad \text{in } \Omega_i, \tag{2.9}$$

$$(A\nabla u)\cdot\mathbf{n} = g, \quad \text{on } \Gamma_N, \tag{2.10}$$

$$u = u_D, \quad \text{on } \Gamma_D, \tag{2.11}$$

$$\frac{\partial u(R,\theta)}{\partial \mathbf{n}} = \mathcal{B}u(R,\theta), \quad \text{on } \Gamma_e, \tag{2.12}$$

which has, other than the usual Dirichlet and Neumann boundary conditions, an implicit artificial boundary condition in differential-integral form (2.12). In addition, in the particular case when $\Gamma_D = \emptyset$, the Dirichlet boundary condition (2.11) must be replaced by the integral boundary condition

$$\int_0^{2\pi} u(R,\theta)d\theta = 0. \tag{2.13}$$

Let

$$V = \begin{cases} \{v \in H^1(\Omega_i) | \int_0^{2\pi} u(R,\theta)d\theta = 0\}, & \text{if } \Gamma_D = \emptyset, \\ \{v \in H^1(\Omega_i) | v = u_D \text{ on } \Gamma_D\}, & \text{otherwise;} \end{cases}$$

$$V_0 = \begin{cases} \{v \in H^1(\Omega_i) | \int_0^{2\pi} u(R,\theta)d\theta = 0\}, & \text{if } \Gamma_D = \emptyset, \\ \{v \in H^1(\Omega_i) | v = 0, \text{ on } \Gamma_D\}, & \text{otherwise.} \end{cases}$$

The only difference between the above two definitions is the boundary condition on Γ_D. Suppose that $g \in L^2(\Gamma_N)$, $u_D \in H^{1/2}(\Gamma_D)$, and suppose that $f \in H^{-1}(\Omega)$ instead of $L^2(\Omega)$ so that we can cover more practical problems, for example, the problem of the stray field energy potential in micromagnetics (see Section 6). Without loss of generality, we may assume that, for some $f_0, f_1, f_2 \in L^2(\Omega_i)$ (see [1]),

$$< v,\ f > = \int_{\Omega_i} \big(f_0(x)v(x) + f_1(x)\partial_1 v + f_2(x)\partial_2 v(x)\big)dx, \quad \forall v \in V_0, \tag{2.14}$$

where f is regarded as an element in the dual space V_0^*.

Then, problem (2.9)-(2.12) has the following weak formulation

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) + b(u,v) = f(v), \quad \forall v \in V_0, \end{cases} \tag{2.15}$$

where

$$a(u,v) = \int_{\Omega_i} A\nabla u \cdot \nabla v dx + \int_{\Omega_i} cuvdx, \tag{2.16}$$

$$b(u,v) = \sum_{n=1}^{\infty} \frac{n}{\pi} \int_0^{2\pi} \int_0^{2\pi} \cos n(\theta - \phi) u(R,\theta) v(R,\phi) d\theta d\phi, \tag{2.17}$$

$$f(v) = \int_{\Omega_i} \big(f_0(x)v(x) + f_1(x)\partial_1 v + f_2(x)\partial_2 v(x)\big)dx + \int_{\Gamma_N} gv\, ds. \tag{2.18}$$

In numerical computations, the infinite summation in the exact artificial boundary condition (2.12) needs to be truncated. This leads to a series of approximate artificial boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = \mathcal{B}_N u \equiv -\frac{1}{\pi R} \sum_{n=1}^{N} n \int_0^{2\pi} u(R,\phi) \cos n(\theta - \phi) d\phi, \quad N = 1, 2, \dots, \tag{2.19}$$

and the corresponding variational problems

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) + b_N(u,v) = f(v), \quad \forall v \in V_0, \end{cases} \tag{2.20}$$

where $b_N(u,v)$ is given by

$$b_N(u,v) = \sum_{n=1}^{N} \frac{n}{\pi} \int_0^{2\pi} \int_0^{2\pi} \cos n(\theta - \phi) u(R,\theta) v(R,\phi) d\theta d\phi. \tag{2.21}$$

It is well known that the symmetric bilinear forms $b_N(\cdot,\cdot)$ are uniformly bounded in $H^1(\Omega_i) \times H^1(\Omega_i)$, i.e., there exists a positive constant C, such that

$$|b_N(u,v)| \le C \|u\|_{1,\Omega_i} \|v\|_{1,\Omega_i}, \quad \forall u, v \in H^1(\Omega_i) \text{ and } \forall N. \tag{2.22}$$

Furthermore, the bilinear forms $a(u,v) + b(u,v)$ and $a(u,v) + b_N(u,v)$ are symmetric, uniformly bounded and V_0-elliptic, and we have the following existence theorem(see [9] and [10]):

Theorem 2.1. *Problem* (2.15) *has a unique solution. For each N, problem* (2.20) *has a unique solution.*

Furthermore, we have the following convergence theorem whose proof could be given from Lemma 4.1 in [10].

Theorem 2.2. *Let u, $u_N \in H^1(\Omega_i)$ be the solutions of* (2.15) *and* (2.20) *respectively. Suppose there exist $R_0 < R$ and an integer $k \geq 1$ such that $\Gamma \subset B(0, R_0)$ and $u|_{\partial B(0,R_0)} \in H^{k-\frac{1}{2}}(\partial B(0, R_0))$. Then, we have*

$$|u - u_N|_{1,\Omega_i} \leq \frac{C}{(N+1)^{k+1}} \left(\frac{R_0}{R}\right)^{N+1} |u|_{k-\frac{1}{2},\partial B(0,R_0)}, \tag{2.23}$$

where C is a constant independent of k, R_0 and N.

Remark 2.1. *In applications, we can always choose R and R_0 so that u is sufficiently smooth near $\partial B(0, R_0)$. Hence, the inequality* (2.23) *indicates that a small N would usually be sufficient to achieve a good approximation.*

3. The Equivalent Mixed Problem

With the method developed in [6] (Section7.1. page 383-386), we establish a mixed problem equivalent to problem (2.20) as follows (see also [4])

$$\begin{cases} \text{Find } (p, u) \in L \times V, \text{ such that} \\ \alpha(p, q) - \beta(q, u) = 0, \qquad \forall q \in L, \\ \beta(p, v) + \gamma(u, v) = f(v), \qquad \forall v \in V_0, \end{cases} \tag{3.1}$$

where $L \equiv (L^2(\Omega_i))^2$ and

$$\alpha(p, q) = \int_{\Omega_i} Ap \cdot q \, dx, \tag{3.2}$$

$$\beta(q, u) = \int_{\Omega_i} Aq \cdot \nabla u \, dx, \tag{3.3}$$

$$\gamma(u, v) = \int_{\Omega_i} c\, uv \, dx + b_N(u, v). \tag{3.4}$$

Using a similar technique as is used in [4] for bounded domain problems with explicit boundary conditions, we establish the following theorem, which is useful in the a posteriori error estimation for the finite element approximations of problem (2.20).

Theorem 3.1. *Let $\mathcal{A} : L \times V_0 \to (L \times V_0)^*$ be an operator defined by $\mathcal{A}(p, u)(q, v) := \mathcal{A}_1(p, u)(q) + \mathcal{A}_2(p, u)(v)$ with*

$$\mathcal{A}_1(p, u)(q) = \alpha(p, q) - \beta(q, u),$$
$$\mathcal{A}_2(p, u)(v) = \beta(p, v) + \gamma(u, v).$$

Then, $\mathcal{A}$ is both surjective and injective, and we have

$$C'\big(\|p\|_0 + \|u\|_1\big) \leq \|\mathcal{A}(p, u)\|_{(L\times V_0)^*} \leq C\big(\|p\|_0 + \|u\|_1\big), \;\; \forall (p, u) \in L \times V_0, \tag{3.5}$$

where $0 < C' \leq C$ are constants independent of p and u.

Proof. We only need to show (3.5). The second inequality follows as a direct consequence from the boundedness of the coefficients A and c, and the uniform boundedness of b_N (see (2.22)). The first inequality is equivalent to the following inf-sup condition:

$$\inf_{y\in L\times V_0}\sup_{z\in L\times V_0}\frac{(\mathcal{A}y,z)}{\|y\|\,\|z\|}\geq C'. \tag{3.6}$$

In fact, for any $y=(p,u)\in L\times V_0$, let $z=(q,v)=(p-\nabla u,2u)\in L\times V_0$. Then it follows from the uniform V_0-ellipticity of $a(u,v)+b_N(u,v)$ and the uniform positive definiteness of the coefficient matrix A that there exists a constant $C'>0$ such that

$$\begin{aligned}\mathcal{A}(p,u)(q,v)&=\alpha(p,q)-\beta(q,u)+\beta(p,v)+\gamma(u,v)\\&=\alpha(p,p)+\beta(\nabla u,u)+2\gamma(u,u)\\&\geq 6C'\big(\|p\|^2_{0,\Omega_i}+\|u\|^2_{1,\Omega_i}\big)\\&\geq C'\big(\|p\|_{0,\Omega_i}+\|u\|_{1,\Omega_i}\big)\big(\|p\|_{0,\Omega_i}+3\|u\|_{1,\Omega_i}\big)\\&\geq C'\big(\|p\|_{0,\Omega_i}+\|u\|_{1,\Omega_i}\big)\big(\|q\|_{0,\Omega_i}+\|v\|_{1,\Omega_i}\big),\end{aligned}$$

which yields (3.6). This completes the proof. □

4. The Finite Element Discretization

Let $\mathcal{T}_h=\{K\}$ be a family of regular triangulations of Ω_i satisfying

$$|a_{ij}-b_{ij}|\leq C\,h_K^2,\quad \forall K \text{ on } \Gamma_e, \tag{4.1}$$

for some constant C, where a_{ij} is the midpoint of the arc $\widehat{a_ia_j}$, and b_{ij} is the midpoint of the section $\overline{a_ia_j}$ (Figure 4.1), and h_K is the diameter of element K, which guarantees that the geometric non-conforming error is of higher order (Section 4.3 in [6]).

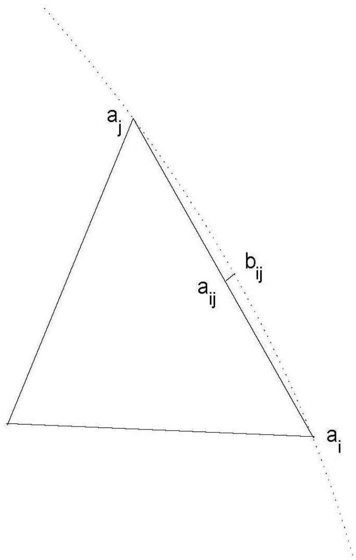

Fig. 4.1. The element near Γ_e

We use the Crouzeix-Raviart element to construct the finite element spaces,

$$U_h = \{v|_K \in P_1(K), \forall K \in \mathcal{T}_h : v \text{ is continuous on } \mathcal{M}_h\},$$

where $\mathcal{M}_h = \{$the midpoints of the edges in $\mathcal{T}_h\}$, and

$$V_h = \begin{cases} \{v \in U_h : \int_{\Gamma_e} v\,dx = 0\}, & \text{if } \Gamma_D = \emptyset, \\ \{v \in U_h : v(a_{ij}) = \{\frac{1}{|E|}\int_E u_D ds, \text{ if } a_{ij} \in \mathcal{M}_h \bigcap E, E \subset \Gamma_D\}\}, & \text{otherwise;} \end{cases}$$

$$V_{h,0} = \begin{cases} \{v \in U_h : \int_{\Gamma_e} v\,dx = 0\}, & \text{if } \Gamma_D = \emptyset, \\ \{v \in U_h : v(a_{ij}) = 0, \text{ if } a_{ij} \in \mathcal{M}_h \bigcap \Gamma_D\}, & \text{otherwise.} \end{cases}$$

We consider the following finite element problem for (2.20):

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) + b_N(u_h, v_h) = f_h(v_h), \quad \forall v_h \in V_{h,0}, \end{cases} \tag{4.2}$$

where

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \left\{ \int_K A\nabla u_h \cdot \nabla v_h dx + \int_K c u_h v_h dx \right\}, \tag{4.3}$$

$$\begin{aligned} b_N(u_h, v_h) =& \sum_{n=1}^{N} \frac{n}{\pi} \int_0^{2\pi}\int_0^{2\pi} \cos n(\theta-\phi) u_h(R,\theta) v_h(R,\phi) d\phi d\theta \\ =& \sum_{n=1}^{N} \frac{n}{\pi} \left(\int_0^{2\pi} u_h(R,\theta)\cos n\theta d\theta \int_0^{2\pi} v_h(R,\phi)\cos n\phi d\phi \right. \\ & \left. + \int_0^{2\pi} u_h(R,\theta)\sin n\theta d\theta \int_0^{2\pi} v_h(R,\phi)\sin n\phi d\phi \right), \end{aligned} \tag{4.4}$$

$$f_h(v_h) = \sum_{K\in\mathcal{T}_h} \left\{ \int_K f_0 v_h + f_1\partial_1 v_h + f_2 \partial_2 v_h dx \right\} + \int_{\Gamma_N} g v_h\, ds. \tag{4.5}$$

In the computation of (4.4), the integrals are calculated by numerical quadrature formulas. For example, by the trapezoidal rule we get

$$\int_0^{2\pi} u_h(R,\theta)\cos n\theta d\theta \approx \sum_{\widehat{a_i a_j}\subset\Gamma_e} \frac{1}{2}|\theta_i - \theta_j| \big(u_h|_{\overline{a_i a_j}}(a_i)\cos(n\theta_i) + u_h|_{\overline{a_i a_j}}(a_j)\cos(n\theta_j) \big),$$

where θ_i and θ_j are the polar angles corresponding to a_i and a_j, respectively. Notice that the error introduced by the numerical quadrature is of higher order, for simplicity, we will ignore its effect in the following analysis.

Similar to problem (2.20), we have

Theorem 4.1. *The finite element problem* (4.2) *has a unique solution.*

The following theorem is important for our a posteriori estimation.

Theorem 4.2. *Let* (p, u) *be the solution of the mixed problem* (3.1). *Then, for any* $p_h \in L_h := \{p_h : p_h|_K = \nabla v_h|_K, \forall K \in \mathcal{T}_h,$ *for some* $v_h \in V_h\}$ *and* $\tilde{u}_h \in V$, *we have*

$$\|p - p_h\|_{0,\Omega_i} + \|u - \tilde{u}_h\|_{1,\Omega_i} \le C\big(\|Res_L(p_h, \tilde{u}_h)\|_{L^*} + \|Res_V(p_h, \tilde{u}_h)\|_{V_0^*} \big), \tag{4.6}$$

where $Res_L(p_h,\tilde{u}_h) \in L^*$ *and* $Res_V(p_h,\tilde{u}_h) \in V_0^*$ *are defined by*

$$Res_L(p_h,\tilde{u}_h)(q) := \alpha(p_h,q) - \beta(q,\tilde{u}_h), \tag{4.7}$$

$$Res_V(p_h,\tilde{u}_h)(v) := -f(v) + \beta(p_h,v) + \gamma(\tilde{u}_h,v). \tag{4.8}$$

Proof. It is easy to verify that

$$Res_L(p_h,\tilde{u}_h)(q) = -(\alpha(p-p_h,q) - \beta(q,u-\tilde{u}_h)),$$
$$Res_V(p_h,\tilde{u}_h)(v) = -(\beta(p-p_h,v) + \gamma(u-\tilde{u}_h,v)).$$

Thus, the conclusion follows as a consequence of Theorem 3.1. □

In the remaining of this section, we will introduce a lemma which is to estimate the error of the approximation for functions in V_h by some functions in V. This lemma is useful for our a posteriori error estimation in next section. For simplicity of analysis and illustration, we suppose hereinafter that the boundary $\Gamma_D \cup \Gamma_N$ is piecewise linear. For general curve boundaries, we may use piecewise linear approximations to get sufficient accuracy ([6]).

We firstly introduce some definitions. Given any function $u_h \in V_h$, we could define a function $\tilde{u}_h \in V$ element-wisely in the following way:

$$\tilde{u}_h|_K = \begin{cases} \tilde{u}_h(a_0)\varphi_0(x) + u_D(x_D)(\varphi_1(x)+\varphi_2(x)) & \text{if some edge } \overline{a_1a_2} \subset \Gamma_D, \\ \sum_{i=0}^{2} \tilde{u}_h(a_i)\varphi_i(x), & \text{otherwise;} \end{cases} \tag{4.9}$$

where $K \in \mathcal{T}_h$, a_i with $i=0,1,2$, are nodes of K, $\varphi_i \in P^1(K)$ such that $\varphi_i(a_j) = \delta_{ij}$ are the order-one finite element base functions on the triangle K, $x_D(x) \in \Gamma_D$ is the intersection of the line a_0x and the interval $\overline{a_1a_2}$ (see Figure 4.2), and $\tilde{u}_h(a_i)$ is given as following

$$\tilde{u}_h(a_i) = \begin{cases} u_D(a_i) & \text{if } a_i \in \Gamma_D, \\ \sum_{K'\in\omega_{a_i}} \frac{|K'|}{|\omega_{a_i}|} u_h|_{K'}(a_i), & \text{otherwise.} \end{cases}$$

Here, we denote $\omega_{a_i} = \cup_{\{K'\in\mathcal{T}_h : a_i \in K'\}} K'$.

The following lemma gives a local estimate for the approximation error $\|u_h - \tilde{u}_h\|$.

Lemma 4.1. *Let* $u_h \in V_h$, $\tilde{u}_h \in V$ *is defined as above, denote* $\mathcal{E} = \{$*the edges in* $\mathcal{T}_h\}$. *Then we have*

$$\|u_h - \tilde{u}_h\|_{i,K} \le \begin{cases} C\Big(\sum\limits_{\substack{E\cap K\neq\emptyset \\ \mathcal{E}\ni E\not\subset\partial\Omega_i}} h_E^{3-2i} \|[\nabla u_h \cdot \mathbf{s}]\|_{0,E}^2 \Big)^{1/2}, & \text{if } K\cap\Gamma_D = \varnothing, \\ C\Big(\sum\limits_{\substack{E\cap K\neq\emptyset \\ \mathcal{E}\ni E\subset\Gamma_D}} \Big(h_E^{3-2i} \Big\| \nabla u_h\cdot\mathbf{s} - \frac{\partial u_D}{\partial \mathbf{s}} \Big\|_{0,E}^2 + h_E^{5-2i} \Big\| \frac{\partial^2 u_D}{\partial \mathbf{s}^2} \Big\|_{0,E}^2 \Big) \\ \quad + \sum\limits_{\substack{E\cap K\neq\emptyset \\ \mathcal{E}\ni E\not\subset\partial\Omega_i}} h_E^{3-2i} \|[\nabla u_h \cdot \mathbf{s}]\|_{0,E}^2 \Big)^{1/2}, & \text{otherwise,} \end{cases}$$

for any $K \in \mathcal{T}_h$ and i=0,1, and

$$\|u_h - \tilde{u}_h\|_{0,E} \le \begin{cases} C\Big(\sum\limits_{\substack{E'\cap E\neq\emptyset \\ \mathcal{E}\ni E\not\subset\partial\Omega_i}} h_{E'}^2 \|[\nabla u_h\cdot \mathbf{s}]\|_{0,E'}^2\Big)^{1/2}, & \text{if } E\cap\Gamma_D=\emptyset, \\ C\Big(\sum\limits_{\substack{E'\cap E\neq\emptyset \\ \mathcal{E}\ni E'\subset\Gamma_D}} \Big(h_{E'}^2 \Big\|\nabla u_h\cdot\mathbf{s} - \dfrac{\partial u_D}{\partial \mathbf{s}}\Big\|_{0,E'}^2 + h_{E'}^4 \Big\|\dfrac{\partial^2 u_D}{\partial \mathbf{s}^2}\Big\|_{0,E'}^2\Big) \\ \quad + \sum\limits_{\substack{E'\cap E\neq\emptyset \\ \mathcal{E}\ni E'\not\subset\partial\Omega_i}} h_{E'}^2 \|[\nabla u_h\cdot\mathbf{s}]\|_{0,E'}^2\Big)^{1/2}, & \text{otherwise}, \end{cases}$$

for any $E \in \mathcal{E}$, where $\mathbf{s}$ denotes the unit tangent vectors to corresponding edges, and $[\cdot]$ is the jump of a function on corresponding edges.

Proof. We only prove the first inequality with $i = 1$ and when the element K has one edge $\overline{a_1a_2}$ such that $\overline{a_1a_2} \subset \Gamma_D$. In this case, we would like to prove

$$\begin{aligned} &\|\nabla u_h - \nabla\tilde{u}_h\|_{0,K} \qquad (4.10) \\ \le& C\Big(h_{\overline{a_1a_2}} \Big\|\nabla u_h\cdot\mathbf{s} - \frac{\partial u_D}{\partial\mathbf{s}}\Big\|_{0,\overline{a_1a_2}}^2 + h_{\overline{a_1a_2}}^3 \Big\|\frac{\partial^2 u_D}{\partial\mathbf{s}^2}\Big\|_{0,\overline{a_1a_2}}^2 + \sum_{\mathcal{E}\ni E\ni a_0} h_E \|[\nabla u_h\cdot\mathbf{s}]\|_{0,E}^2\Big)^{1/2}, \end{aligned}$$

where a_0 is another node of K. The proof of other cases can be done in a similar way.

For such an element K, from the definition (4.9) of $\tilde{u}_h$ and also noticed that $u_h(x)$ can be rewritten as

$$u_h(x) = u_h(a_0)\varphi(x) + u_h(x_D)(\varphi_1(x) + \varphi_2(x)),$$

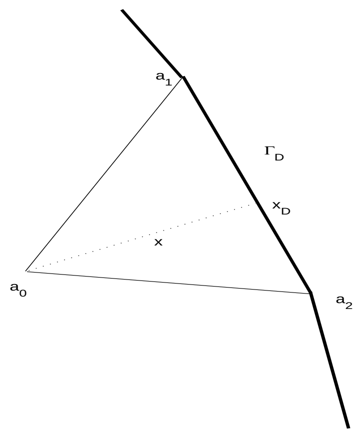

Fig. 4.2. The element near Γ_D

we have

$$\begin{aligned}\int_K |\nabla u_h - \nabla \tilde{u}_h|^2 dx \le & 2\Big(\big(u_h|_K(a_0) - \tilde{u}_h(a_0)\big)^2 \int_K |\nabla \varphi_0|^2 dx \\ & + \int_K \big(u_D(x_D) - u_h(x_D)\big)^2 |\nabla\big(\varphi_1(x) - \varphi_2(x)\big)|^2 dx \\ & + \int_K \big(\varphi_1(x) - \varphi_2(x)\big)^2 |\nabla\big(u_D(x_D) - u_h(x_D)\big)|^2 dx\Big) \\ = & 2(I_1 + I_2 + I_3). \end{aligned} \tag{4.11}$$

For I_1, we have

$$\begin{aligned} I_1 \le & C\big(u_h|_K(a_0) - \tilde{u}_h(a_0)\big)^2 \\ \le & C\Big(u_h|_K(a_0) - \sum_{K' \in \omega_{a_0}} \frac{|K'|}{|\omega_{a_0}|} u_h|_{K'}(a_0)\Big)^2 \\ = & C\Big(\sum_{K' \in \omega_{a_0}} \frac{|K'|}{|\omega_{a_0}|} \big(u_h|_K(a_0) - u_h|_{K'}(a_0)\big)\Big)^2 \\ \le & C\Big(\sum_{\substack{K_i', K_j' \in \omega_{a_0} \\ K_i' \cap K_j' = E \in \mathcal{E}}} \big|u_h|_{K_i'}(a_0) - u_h|_{K_j'}(a_0)\big|\Big)^2 \\ \le & C\left(\sum_{\mathcal{E} \ni E \ni a_0} \frac{1}{2}\left|\int_E [\nabla u_h \cdot \mathbf{s}] ds\right|\right)^2 \le C\left(\sum_{\mathcal{E} \ni E \ni a_0} h_E \int_E |[\nabla u_h \cdot \mathbf{s}]|^2 \, ds\right). \end{aligned} \tag{4.12}$$

For I_2 and I_3, we let $\hat{K} = \{\hat{x} \in R^2 : 0 \le \hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \le 1\}$ be the reference element, $\hat{a}_0 = (1, 0)$, $\hat{a}_1 = (0, 1)$ and $\hat{a}_2 = (0, 0)$ be the nodes of $\hat{K}$, and $\hat{\varphi}_i \in P^1(\hat{K})$ such that $\hat{\varphi}_i(\hat{a}_j) = \delta_{ij}$ be the finite element base functions in $\hat{K}$. Then by classical scaling techniques in finite element analysis ([6]) and also by the definition of the boundary conditions for the finite element space V_h, we get

$$\begin{aligned} I_2 \le & C \int_{\hat{K}} |u_D(\hat{x}_D) - u_h(\hat{x}_D)|^2 \; |\nabla_{\hat{x}}\big(\hat{\varphi}_1(\hat{x}) - \hat{\varphi}_2(\hat{x})\big)|^2 d\hat{x} \\ \le & C \int_{\hat{K}} |u_D(\hat{x}_D) - u_h(\hat{x}_D)|^2 d\hat{x} \\ \le & C \int_{\overline{\hat{a}_1 \hat{a}_2}} |u_D(\hat{s}) - u_h(\hat{s})|^2 d\hat{s} \le \frac{C}{h_{\overline{a_1 a_2}}} \int_{\overline{a_1 a_2}} |u_D(s) - u_h(s)|^2 ds \\ \le & C\left(h_{\overline{a_1 a_2}} \int_{\overline{a_1 a_2}} \left(\frac{\partial u_D}{\partial s} - \nabla u_h \cdot \mathbf{s}\right)^2 ds + h^3_{\overline{a_1 a_2}} \int_{\overline{a_1 a_2}} \left(\frac{\partial^2 u_D}{\partial s^2}\right)^2 ds\right), \end{aligned} \tag{4.13}$$

and

$$
\begin{aligned}
I_3 &\le C \int_{\hat{K}} |\hat{\varphi}_1(\hat{x}) - \hat{\varphi}_2(\hat{x})|^2 \, |\nabla_{\hat{x}}\big(u_D(\hat{x}_D) - u_h(\hat{x}_D)\big)|^2 d\hat{x} \\
&= C \int_{\hat{K}} |\hat{\varphi}_1(\hat{x}) - \hat{\varphi}_2(\hat{x})|^2 \, |\frac{\partial\big(u_D(\hat{x}_D) - u_h(\hat{x}_D)\big)}{\partial \hat{x}_2}|^2 \Big(1 + \frac{\hat{x_2}^2}{(1-\hat{x}_1)^2}\Big) d\hat{x} \\
&\le C \int_{\hat{K}} |\hat{\varphi}_1(\hat{x}) - \hat{\varphi}_2(\hat{x})|^2 \, |\frac{\partial\big(u_D(\hat{x}_D) - u_h(\hat{x}_D)\big)}{\partial \hat{x}_2}|^2 d\hat{x} \\
&\le C \int_{\overline{\hat{a}_1\hat{a}_2}} |\frac{\partial\big(u_D(\hat{x}_D) - u_h(\hat{x}_D)\big)}{\partial \hat{x}_2}|^2 d\hat{x}_2 \\
&= C \int_{\overline{\hat{a}_1\hat{a}_2}} \left| \frac{\partial u_D}{\partial \hat{s}} - \nabla u_h \cdot \hat{\mathbf{s}} \right|^2 d\hat{s} \\
&\le C h_{\overline{a_1a_2}} \int_{\overline{a_1a_2}} \left| \frac{\partial u_D}{\partial s} - \nabla u_h \cdot \mathbf{s} \right|^2 ds. \qquad (4.14)
\end{aligned}
$$

From the inequalities (4.11)-(4.14), we get (4.10). □

5. The a Posteriori Estimator and Its Reliability and Efficiency

For $K \in \mathcal{T}_h$, let $\mathcal{E}_K = \{E \subset \mathbf{R}^2 : E \text{ is an edge of } K\}$, and define

$$
\begin{aligned}
\eta_K^2 = &\sum_{\mathcal{E}_K \ni E \not\subset \partial\Omega_i} h_E \left(\|[(A\nabla_h u_h + \bar{\mathbf{f}}) \cdot \mathbf{n}]\|_{0,E}^2 + \|[\nabla_h u_h \cdot \mathbf{s}]\|_{0,E}^2 \right) \\
&+ \sum_{\mathcal{E}_K \ni E \subset \Gamma_D} h_E \left\| \nabla_h u_h \cdot \mathbf{s} - \frac{\partial u_D}{\partial \mathbf{s}} \right\|_{0,E}^2 + \sum_{\mathcal{E}_K \ni E \subset \Gamma_N} h_E \|A\nabla_h u_h \cdot \mathbf{n} - g\|_{0,E}^2 \\
&+ \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E \left\| \frac{\partial u_h}{\partial \mathbf{n}} - \mathcal{B}_N u_h \right\|_{0,E}^2, \qquad (5.1)
\end{aligned}
$$

where $\nabla_h u_h \in (L^2(\Omega_i))^2$ is defined by $(\nabla_h u_h)|_K := \nabla(u_h|_K)$, $\mathbf{n}$ and $\mathbf{s}$ denote the unit normal and tangent vectors to the corresponding edges respectively, and where $\bar{\mathbf{f}} = (-f_1, -f_2)^T$ with f_1, f_2 being given in (2.18). Suppose that $f \in H^{-1}(\Omega)$ satisfies a further assumption that $\partial_1(f_1|_K)$, $\partial_2(f_2|_K) \in L^2(K)$, for all $K \in \mathcal{T}_h$, and $A \in (H^1(\Omega))^{2\times 2}$. Then, we have the following a posteriori error estimate.

Theorem 5.1. *Let u be the solution of problem* (2.20)*, and u_h be the solution of the finite element problem* (4.2)*. Then, we have*

$$
\begin{aligned}
&\|\nabla u - \nabla_h u_h\|_{0,\Omega_i} \\
\le &C\Big(\sum_{K \in \mathcal{T}_h} \big(h_K^2 \|\mathcal{R}_K(u_h)\|_{0,K}^2 + \eta_K^2\big) + \sum_{\mathcal{E} \ni E \subset \Gamma_D} h_E^3 \int_E |\frac{\partial^2 u_D}{\partial \mathbf{s}^2}|^2 ds \Big)^{\frac{1}{2}},
\end{aligned}
$$

where

$$\mathcal{R}_K(u_h) := (\hat{f} + \mathrm{div}(A\nabla_h u_h) - c u_h)|_K$$

with $\hat{f}|_K = f_0 - \partial_1(f_1|_K) - \partial_2(f_2|_K)$.

Proof. Let $p = \nabla u$. Then (p, u) is the solution of the mixed problem (3.1). Thus, it follows from Theorem 4.2 that, for $p_h = \nabla_h u_h$ and $\tilde{u}_h \in V$ defined by (4.9),

$$\begin{aligned}\|\nabla u - \nabla_h u_h\|_{0,\Omega_i} \le& \|p - p_h\|_{0,\Omega_i} + \|u - \tilde{u}_h\|_{1,\Omega_i}\\ \le& C\left(\|Res_L(p_h, \tilde{u}_h)\|_{L^*} + \|Res_V(p_h, \tilde{u}_h)\|_{V_0^*}\right).\end{aligned} \tag{5.2}$$

Note that

$$\begin{aligned}\|Res_L(p_h, \tilde{u}_h)\|_{L^*} =& \sup_{q \in L} \frac{|\alpha(p_h, q) - \beta(q, \tilde{u}_h)|}{\|q\|_L}\\ \le& \|A\nabla_h u_h - A\nabla \tilde{u}_h\|_{0,\Omega_i} \le C\|\nabla_h u_h - \nabla \tilde{u}_h\|_{0,\Omega_i}.\end{aligned}$$

Consequently, it follows from Lemma 4.1 that

$$\begin{aligned}\|Res_L(p_h, \tilde{u}_h)\|_{L^*} \le& C\Bigg\{\sum_{\mathcal{E}\ni E\subset\Gamma_D}\left(h_E\left\|\nabla_h u_h \cdot \mathbf{s} - \frac{\partial u_D}{\partial \mathbf{s}}\right\|_{0,E}^2 + h_E^3\left\|\frac{\partial^2 u_D}{\partial \mathbf{s}^2}\right\|_{0,E}^2\right)\\ &+ \sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i} h_E\|[\nabla_h u_h \cdot \mathbf{s}]\|_{0,E}^2\Bigg\}^{\frac{1}{2}}.\end{aligned} \tag{5.3}$$

Next, we consider

$$\|Res_V(p_h, \tilde{u}_h)(v)\|_{V_0^*} = \sup_{v \in V_0} \frac{|-f(v) + \beta(p_h, v) + \gamma(\tilde{u}_h, v)|}{\|v\|_{1,\Omega_i}}. \tag{5.4}$$

Let v_h be the Clément interpolation of v in the conforming finite element space $\tilde{V}_{h,0} \subset V_{h,0}$. Noticing that u_h is the finite element solution of (3.1), we have

$$\begin{aligned}&-f(v) + \beta(p_h, v) + \gamma(\tilde{u}_h, v)\\ =& -\sum_{K\in\mathcal{T}_h}\int_K \big(f_0(v - v_h) + f_1\partial_1(v - v_h) + f_2\partial_2(v - v_h)\big)\,dx\\ &- \int_{\Gamma_N} g(v - v_h)ds + \sum_{K\in\mathcal{T}_h}\int_K A\nabla_h u_h \cdot \nabla_h(v - v_h)\,dx\\ &+ \int_{\Omega_i} c\tilde{u}_h v dx + b_N(\tilde{u}_h, v) - \int_{\Omega_i} c u_h v_h dx - b_N(u_h, v_h)\\ =& -\sum_{K\in\mathcal{T}_h}\int_K \left(\hat{f} + \mathrm{div}(A\nabla_h u_h) + cu_h\right)(v - v_h)dx\\ &+ \int_{\Omega_i} c(\tilde{u}_h - u_h)v dx + \sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i}\int_E [(A\nabla_h u_h + \bar{\mathbf{f}}) \cdot \mathbf{n}](v - v_h)ds\\ &+ \sum_{\mathcal{E}\ni E\subset\Gamma_N}\int_E (A\nabla_h u_h \cdot \mathbf{n} - g)(v - v_h)ds - \int_{\Gamma_e}(\mathcal{B}_N\tilde{u}_h - \mathcal{B}_N u_h)v ds\\ &+ \sum_{\mathcal{E}\ni E\subset\Gamma_e}\int_E (\nabla u_h \cdot \mathbf{n} - \mathcal{B}_N u_h)(v - v_h)ds.\end{aligned} \tag{5.5}$$

By the standard interpolation theory for Sobolev functions [6], we have

$$\begin{aligned}
&\left|\sum_{K\in\mathcal{T}_h} h_K \int_K \left(\hat{f} + \mathrm{div}(A\nabla_h u_h) + cu_h\right)(v - v_h)dx\right| \\
\leq & C\left(\sum_{K\in\mathcal{T}_h} h_K \|\hat{f} + \mathrm{div}(A\nabla_h u_h) + cu_h\|_{0,K}\|v\|_{1,\omega_K}\right) \\
\leq & C\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\hat{f} + \mathrm{div}(A\nabla_h u_h) + cu_h\|_{0,K}^2\right)^{1/2} \|v\|_{1,\Omega_i},
\end{aligned} \tag{5.6}$$

where $\omega_K = \bigcup_{\{K'\in\mathcal{T}_h : K'\cap K\neq\emptyset\}} K'$,

$$\begin{aligned}
&\left|\sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i} \int_E [(A\nabla_h u_h + \bar{\mathbf{f}})\cdot\mathbf{n}](v - v_h)dx\right| \\
\leq & C\left(\sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i} h_E^{1/2}\|[(A\nabla_h u_h + \bar{\mathbf{f}})\cdot\mathbf{n}]\|_{0,E}\|v\|_{1,\omega_E}\right) \\
\leq & C\left(\sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i} h_E\|[(A\nabla_h u_h + \bar{\mathbf{f}})\cdot\mathbf{n}]\|_{0,E}^2\right)^{1/2} \|v\|_{1,\Omega_i},
\end{aligned} \tag{5.7}$$

where $\omega_E = \bigcup_{\{K\in\mathcal{T}_h : E\subset K\}} K$. Similarly, we have

$$\begin{aligned}
&\left|\sum_{\mathcal{E}\ni E\subset\Gamma_N} \int_E (A\nabla_h u_h\cdot\mathbf{n} - g)(v - v_h)ds\right| \\
\leq & C\left(\sum_{\mathcal{E}\ni E\subset\Gamma_N} h_E\|A\nabla_h u_h\cdot\mathbf{n} - g\|_{0,E}^2\right)^{1/2} \|v\|_{1,\Omega_i},
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\left|\sum_{\mathcal{E}\ni E\subset\Gamma_e} \int_E (\nabla u_h\cdot\mathbf{n} - \mathcal{B}_N u_h)(v - v_h)ds\right| \\
\leq & C\left(\sum_{\mathcal{E}\ni E\subset\Gamma_e} h_E\|\nabla u_h\cdot\mathbf{n} - \mathcal{B}_N u_h\|_{0,E}^2\right)^{1/2} \|v\|_{1,\Omega_i}.
\end{aligned} \tag{5.9}$$

It follows from Lemma 4.1 that

$$\begin{aligned}
&\left|\int_{\Omega_i} c(\tilde{u}_h - u_h)v dx\right| \leq C\|\tilde{u}_h - u_h\|_{0,\Omega_i}\|v\|_{0,\Omega_i} \\
\leq & C\Bigg\{\sum_{\mathcal{E}\ni E\subset\Gamma_D}\left(h_E^3\left\|\nabla_h u_h\cdot\mathbf{s} - \frac{\partial u_D}{\partial\mathbf{s}}\right\|_{0,E}^2 + h_E^5\|\frac{\partial^2 u_D}{\partial\mathbf{s}^2}\|_{0,E}^2\right) \\
& + \sum_{\mathcal{E}\ni E\not\subset\partial\Omega_i} h_E^3\|[\nabla_h u_h\cdot\mathbf{s}]\|_{0,E}^2\Bigg\}^{\frac{1}{2}} \|v\|_{0,\Omega_i}.
\end{aligned} \tag{5.10}$$

By Parseval's relation for Fourier transformations, by Lemma 4.1 and also by the trace theorem for the Sobolev space, we have

$$
\begin{aligned}
&\left|\int_{\Gamma_e} (\mathcal{B}_N \tilde{u}_h - \mathcal{B}_N u_h) v \, ds\right| \\
=&\left|\sum_{n=1}^{N} \frac{n}{\pi} \int_0^{2\pi} \int_0^{2\pi R} (\tilde{u}_h - u_h)(R,\theta) v(R,\varphi) \cos n(\theta - \varphi) d\theta d\varphi\right| \\
\leq& C \left(\int_{\Gamma_e} (\tilde{u}_h - u_h)^2 ds\right)^{1/2} \left(\int_{\Gamma_e} v^2 ds\right)^{1/2} \\
\leq& C\Big(\sum_{\substack{E\cap\Gamma_e \neq \emptyset \\ \mathcal{E}\ni E \not\subset \Gamma_e}} h_E^2 \|[\nabla_h u_h \cdot \mathbf{s}]\|_{0,E}^2\Big)^{1/2} \|v\|_{1,\Omega_i}.
\end{aligned}
\tag{5.11}
$$

The proof of the theorem is completed by combining (5.2) with the inequalities (5.3)-(5.11). □

Remark 5.1. Since the term $h_K^2\|\mathcal{R}_K(u_h)\|_{0,K}^2$ is in general a higher order term with respect to η_K^2, Theorem 5.1 implies that η_K is a reliable a posteriori error estimator.

For the efficiency of the a posteriori error estimator, we have the following result.

Theorem 5.2. *Let u be the solution of problem* (2.20)*, u_h be the solution of the finite element problem* (4.2)*, and let the a posteriori error estimator η_K be given by* (5.1)*. Then, for all $K \in \mathcal{T}_h$, we have*

$$
\begin{aligned}
\eta_K \leq& C\Big\{(\|u-u_h\|_{1,\omega_K} + \sum_{\mathcal{E}_K \ni E \subset \Gamma_N} h_E^{1/2}\|g - g_E\|_{0,E} + h_K\|\hat{f} - \hat{f}_K\|_{0,\omega_K} \\
&+ \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2}\|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E} + \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2}\left\|\frac{\partial u}{\partial \mathbf{n}} - \left(\frac{\partial u}{\partial \mathbf{n}}\right)_E\right\|_{0,E}\Big\},
\end{aligned}
\tag{5.12}
$$

where $g_E = \frac{1}{|E|}\int_E g ds$, $\hat{f}_K = \frac{1}{|K|}\int_K \hat{f}(x)dx$ and $\left(\frac{\partial u}{\partial \mathbf{n}}\right)_E = \frac{1}{|E|}\int_E \frac{\partial u}{\partial \mathbf{n}} ds$.

Proof. Except for the term

$$
\sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2}\left\|\frac{\partial u_h}{\partial \mathbf{n}} - \mathcal{B}_N u_h\right\|_{0,E} \quad \textit{in } \eta_K,
$$

which, as to be shown below, leads to additional terms

$$
\sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2}\|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E} + \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2}\|\frac{\partial u}{\partial \mathbf{n}} - \left(\frac{\partial u}{\partial \mathbf{n}}\right)_E\|_{0,E}
$$

on the right hand side of (5.12), the estimates for all of the other terms in η_K are standard [4,12]. As in [12], let $\hat{K} = \{\hat{x} \in R^2 : 0 \leq \hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \leq 1\}$ be the reference finite element with $\hat{E} = \hat{K} \cap \{\hat{x}_2 = 0\}$, let $F_K : \hat{K} \to K \subset \omega_E$ be the one-to-one quadratic mapping with $F_K(\hat{E}) = E \subset \Gamma_e$ (see Section 4), and let $\hat{\mathcal{P}} : C(\hat{E}) \to C(\hat{K})$ be given by $\hat{\mathcal{P}}(\hat{v})(\hat{x}) = \hat{v}(\hat{x}_1)$. We define an extension operator $\mathcal{P} : C(E) \to C(\omega_E)$ by $(\mathcal{P}v)|_K := (\hat{\mathcal{P}}(v \circ F_K)) \circ F_K^{-1}$. Denote

$$
P_h = \mathcal{P}(\frac{\partial u_h}{\partial \mathbf{n}} - (\mathcal{B}_N u_h)_E), \quad \textit{where} \ (\mathcal{B}_N u_h)_E = \frac{1}{|E|}\int_E \mathcal{B}_N u_h ds,
$$

and let b_E be the edge-bubble function defined on ω_E with respect to the edge E [12]. Then, recall that $\mathcal{B}_N u = \frac{\partial u}{\partial \mathbf{n}}$, we have

$$\begin{aligned}
&\left\| \frac{\partial u_h}{\partial \mathbf{n}} - (\mathcal{B}_N u_h)_E \right\|_{0,E}^2 \\
\le & C \int_E b_E P_h \left(\frac{\partial (u_h - u)}{\partial \mathbf{n}} + \mathcal{B}_N u - (\mathcal{B}_N u_h)_E \right) ds \\
= & C \left(\int_{K \supset E} \nabla (b_E P_h) \nabla (u_h - u) dx + \int_E (b_E P_h)(\mathcal{B}_N u - (\mathcal{B}_N u_h)_E) ds \right) \\
\le & C \left(\|\nabla u - \nabla_h u_h\|_{0,K} \|\nabla (b_E P_h)\|_{0,K} + \|\mathcal{B}_N u - (\mathcal{B}_N u_h)_E\|_{0,E} \|b_E P_h\|_{0,E} \right) \\
\le & C \left(h_E^{-1} \|\nabla u - \nabla_h u_h\|_{0,K} \|b_E P_h\|_{0,K} + \|\mathcal{B}_N u - (\mathcal{B}_N u_h)_E\|_{0,E} \|b_E P_h\|_{0,E} \right) \\
\le & C \left(h_E^{-1/2} \|\nabla u - \nabla_h u_h\|_{0,K} + \|\mathcal{B}_N u - (\mathcal{B}_N u_h)_E\|_{0,E} \right) \left\| \frac{\partial u_h}{\partial \mathbf{n}} - (\mathcal{B}_N u_h)_E \right\|_{0,E}.
\end{aligned}$$

This gives

$$\begin{aligned}
& h_E^{1/2} \| \frac{\partial u_h}{\partial \mathbf{n}} - \mathcal{B}_N u_h \|_{0,E} \qquad (5.13)\\
\le & h_E^{1/2} \left(\| \frac{\partial u_h}{\partial \mathbf{n}} - (\mathcal{B}_N u_h)_E \|_{0,E} + \|\mathcal{B}_N u_h - (\mathcal{B}_N u_h)_E\|_{0,E} \right) \\
\le & \|\nabla u - \nabla_h u_h\|_{0,K} + h_E^{1/2} \left(\|\mathcal{B}_N u - (\mathcal{B}_N u_h)_E\|_{0,E} + \|\mathcal{B}_N u_h - (\mathcal{B}_N u_h)_E\|_{0,E} \right) \\
\le & \|\nabla u - \nabla_h u_h\|_{0,K} + h_E^{1/2} \left(\|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E} + 2\|\mathcal{B}_N u_h - (\mathcal{B}_N u_h)_E\|_{0,E} \right).
\end{aligned}$$

Since

$$\frac{\partial u}{\partial \mathbf{n}} = \mathcal{B}_N u, \quad \|(\mathcal{B}_N u)_E - (\mathcal{B}_N u_h)_E\|_{0,E} \le \|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E},$$

we have

$$\begin{aligned}
& \|\mathcal{B}_N u_h - (\mathcal{B}_N u_h)_E\|_{0,E} \\
\le & \|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E} + \|(\mathcal{B}_N u)_E - (\mathcal{B}_N u_h)_E\|_{0,E} + \|\mathcal{B}_N u - (\mathcal{B}_N u)_E\|_{0,E} \\
\le & 2\|\mathcal{B}_N u - \mathcal{B}_N u_h\|_{0,E} + \left\| \frac{\partial u}{\partial \mathbf{n}} - \left(\frac{\partial u}{\partial \mathbf{n}} \right)_E \right\|_{0,E}. \qquad (5.14)
\end{aligned}$$

The proof is completed by combining (5.13) and (5.14). □

Remark 5.2. If $\partial K \cap \Gamma_e = \emptyset$, then the right hand side of (5.12) reduces to

$$C \left(\|u - u_h\|_{1,\omega_K} + h_K \|\hat{f} - \hat{f}_K\|_{0,\omega_K} + \sum_{\mathcal{E}_K \ni E \subset \Gamma_N} h_E^{1/2} \|g - g_E\|_{0,E} \right).$$

This implies that, for the elements not lying on the artificial boundary Γ_e, our estimator is efficient.

Remark 5.3. On the artificial boundary Γ_e, we have

$$\begin{aligned}
& \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{1/2} \left\| \frac{\partial u}{\partial \mathbf{n}} - \left(\frac{\partial u}{\partial \mathbf{n}} \right)_E \right\|_{0,E} \\
\le & C \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^{\frac{3}{2}} \left\| \frac{\partial^2 u}{\partial \mathbf{s} \partial \mathbf{n}} \right\|_{0,E} \le C \sum_{\mathcal{E}_K \ni E \subset \Gamma_e} h_E^2 |u|_{2,\infty,K}, \qquad (5.15)
\end{aligned}$$

which is of the same order as the term $\|u-u_h\|_{1,\omega_K} \le Ch_K^2|u|_{2,\infty,K}$. Furthermore, the last term of (5.12) can be replaced by a higher order term under the assumption of better regularity of u near Γ_e, which is true in most cases. On second last term of (5.12) noticing that

$$\left|\sum_{n=1}^{N} n\int_0^{2\pi}(u-u_h)\cos n(\theta-\phi)d\theta\right| \le \sqrt{2\pi}N^2R^{-1}\left(\int_{\Gamma_e}(u-u_h)^2ds\right)^{1/2},$$

we have

$$\begin{aligned}&\sum_{\mathcal{E}_K\ni E\subset\Gamma_e} h_E^{1/2}\|\mathcal{B}_N u-\mathcal{B}_N u_h\|_{0,E} \le CN^2h_E\left(\int_{\Gamma_e}(u-u_h)^2ds\right)^{1/2}\\ \le&CN^2h_E^{3/2}\sum_{K'\cap\Gamma_e\neq\emptyset}\|u-u_h\|_{1,K'} \le CN^2h_E^{3/2}\sum_{K'\cap\Gamma_e\neq\emptyset}h_{K'}^2|u|_{2,\infty,K'}.\end{aligned} \tag{5.16}$$

When $\partial K\cap\Gamma_e\neq\emptyset$, the right hand side of (5.16) is of order $\mathcal{O}(h_E^2)$, which is of the same order as the term $\|u-u_h\|_{1,\omega_K}\le Ch_K^2|u|_{2,\infty,K}$, provided that $\max_{E\subset\Gamma_e}\{h_E\}\le(\min_{E\subset\Gamma_e}\{h_E\})^{1/2}$. So, as long as the condition $\max_{E\subset\Gamma_e}\{h_E\}\le(\min_{E\subset\Gamma_e}\{h_E\})^{1/2}$ is satisfied, our a posteriori error estimator is also efficient on the artificial boundary. In fact, the condition is easily satisfied if the solution u is sufficiently smooth near the artificial boundary, which is always the case for our problem if R is sufficiently large. The violation of the condition implies that the solution has certain singularity on the artificial boundary. In the later case, a larger circle should be chosen as a new artificial boundary.

6. An Adaptive Algorithm and Numerical Examples

In this section, we apply a standard mesh adaptive algorithm with η_K defined by (5.1) as the a posteriori error estimator, to some typical numerical examples. In addition, it is well-known that the recovery technique can give much efficient a posteriori error estimators in many numerical examples. Thus, we would also like to make some numerical comparisons between our error estimator and some recovery type error estimators in our numerical experiments.

Algorithm:

Step 1. Given an initial mesh $\mathcal{T}_0$, a tolerance $TOL>0$, and a number $0<\mu<1$.

Step 2. Solve the problem on the mesh $\mathcal{T}_i$, to obtain the solution u_h.

Step 3. Calculate the error estimator η_K on each element K, and set $\eta_{max}=\max_{K\in\mathcal{T}}\eta_K$. If $(\sum_{K\in\mathcal{T}}\eta_K^2)^{1/2}<TOL$, stop, else go to the next step.

Step 4. If $\eta_K>\mu\eta_{max}$, mark it. Generate a new mesh by regularly refining the marked elements, that is to divide triangles into four by joining the midpoints of edges.

Step 5. Refine further the other elements (red-green-blue refinement, p.108 in [12]) to eliminate the hanging nodes. Go back to Step 2.

Example 1. Consider an exterior problem for the Poisson equation. We consider the following problem defined on $\Omega=\{(r,\theta): r>0.5,\ 0\le\theta\le2\pi\}$, i.e., a planar domain outside

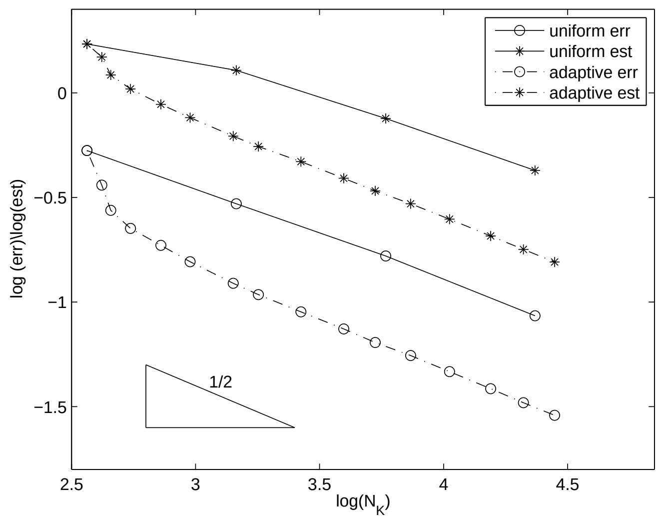

Fig. 6.1. Convergence behavior for Example 1.

a circular obstacle of radius $r_0 = 0.5$:

$$-\Delta u = 0, \qquad \text{in } \Omega, \tag{6.1a}$$

$$u(0.5,\theta) = 0.5\ln\left(\frac{0.4525 + 0.45\sin\theta}{0.4525 - 0.45\sin\theta}\right), \qquad 0 \le \theta \le 2\pi, \tag{6.1b}$$

$$u \to 0, \qquad r \to \infty. \tag{6.1c}$$

The problem has an exact solution:

$$u(r,\theta) = 0.5\ln\left(\frac{r^2 + 0.9r\sin\theta + 0.45^2}{r^2 - 0.9r\sin\theta + 0.45^2}\right). \tag{6.2}$$

The numerical results of corresponding finite element problem (4.2) with $N = 9$ are shown in Table 6.1 and Figure 6.1, where N_K is the number of elements in a mesh, $err = \frac{\|\nabla u - \nabla_h u_h\|_{0,\Omega_i}}{\|\nabla u\|_{0,\Omega_i}}$ is the relative error of the numerical solution in $H^1(\Omega_i)$ with semi-norm for $R = 2$, and est is the corresponding error estimate given by our a posteriori error estimator which is also divided by $\|\nabla u\|_{0,\Omega_i}$.

It is clearly shown in Table 6.1 that the ratios of the estimates and the errors converge to a constant during the process of adaptivity, which verifies that our posteriori estimator is both reliable and efficient, and suggests further that, the estimator, multiplied by a constant factor is asymptotically exact in this situation. Compare the convergence behavior of the errors and estimates of the numerical solutions on both the uniformly refined and adaptively refined meshes shown in Figure 6.1, we see that the adaptive method produces much sharper numerical results and reaches the optimal convergence order $O(N_K^{-1/2})$.

To compare our a posteriori error estimator with some recovery type error estimators, we also compute the problem by using the average type error estimator $\tilde{\eta}_K = \|\nabla u_h - G_h\nabla u_h\|_{0,K}$ ([15]), where $G_h\nabla u_h|_K \in (P^1(K))^2$ with

$$G_h\nabla u_h(a_i) = \sum_{K'\in\omega_{a_i}} \frac{|K'|}{|\omega_{a_i}|}\nabla u_h|_{K'}(a_i),$$

Table 6.1: Numerical results for Example 1 using estimator η.

N_K	Error err	Estimator est	err/est
547	0.224978	1.044764	0.21534
951	0.155840	0.761614	0.20462
1419	0.122980	0.620880	0.19807
2659	0.089849	0.469365	0.19143
5301	0.064120	0.340269	0.18844
7368	0.055490	0.294950	0.18813
10567	0.046520	0.249202	0.18668
15484	0.038554	0.207170	0.18610

Table 6.2: Numerical results for Example 1 using estimator $\tilde{\eta}$.

N_K	Error err	Estimator est_A	err/est_A
517	0.240136	0.224495	1.06967
869	0.163397	0.157478	1.03759
1510	0.118884	0.115879	1.02593
2602	0.090941	0.088565	1.02683
5127	0.065583	0.063895	1.02643
7946	0.052668	0.051321	1.02623
10960	0.045165	0.043993	1.02664
13638	0.040550	0.039534	1.02569

for any node a_i of K. Table 6.2 is the corresponding numerical results where err has the same meaning as before and est_A is the corresponding error estimate given by the estimator $\tilde{\eta}$, which is also divided by $\|\nabla u\|_{0,\Omega_i}$. It shows that the two estimators could generate almost the same accuracy on the meshes with similar numbers of freedoms while the ratio err/est_A is close to 1.

Example 2. Consider the potential of the stray field energy in micromagnetics. In micromagnetic simulations, one difficulty is the non-locality of the stray field. For a ferromagnetic material occupying a bounded domain $\Omega_m \subset \mathbf{R}^d$, the potential of the stray field energy u is determined by the magnetization field $\mathbf{m}$ through the following Maxwell's equation:

$$\text{div}(-\nabla u + \mathbf{m}\chi_{\Omega_m}) = 0, \qquad \text{in } \mathbf{R}^d, \tag{6.3a}$$
$$u \to 0, \qquad \text{as } |x| \to \infty. \tag{6.3b}$$

We notice here that Eq. (6.3a) is considered to hold in $H^{-1}(\mathbf{R}^d)$, and thus, in two-dimensions, the problem is equivalent to problem (2.1)-(2.4) with $\Omega = \mathbf{R}^2$, $\Gamma = \Gamma_N \bigcup \Gamma_D = \emptyset$, and f being such that $f_0 = 0$, $-\bar{\mathbf{f}} = (f_1, f_2)^T = \mathbf{m}\chi_{\Omega_m}$ (see (2.14)), where χ_{Ω_m} is the characteristic function of Ω_m.

In our numerical experiments, we take $\Omega_m = [-0.1, 0.1] \times [-0.5, 0.5]$ and $\Omega_i = B(0,1)$, and set $\mathbf{m} = (\cos(20\pi xy), \sin(20\pi xy))^T$, which are unit vectors in $\mathbf{R}^2$. Moreover, we set $N = 9$. Figure 6.2 shows the a posteriori error estimates for the numerical solutions on both the uniformly and adaptively refined meshes. It can be observed that the expected optimal convergence rate $\mathcal{O}(N_K^{-1/2})$ is obtained by the adaptive method. In Figure 6.3, we show the initial mesh and some selected adaptively refined meshes, and it is clearly seen that the area where u has significantly larger variance of derivatives are refined during the adaptivity process.

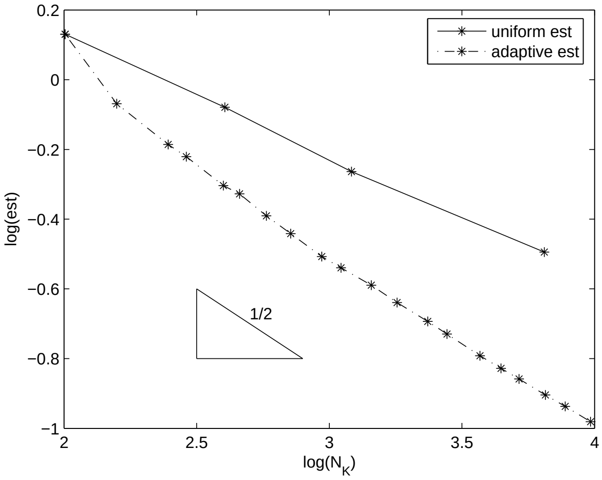

Fig. 6.2. The convergence behavior for Example 2.

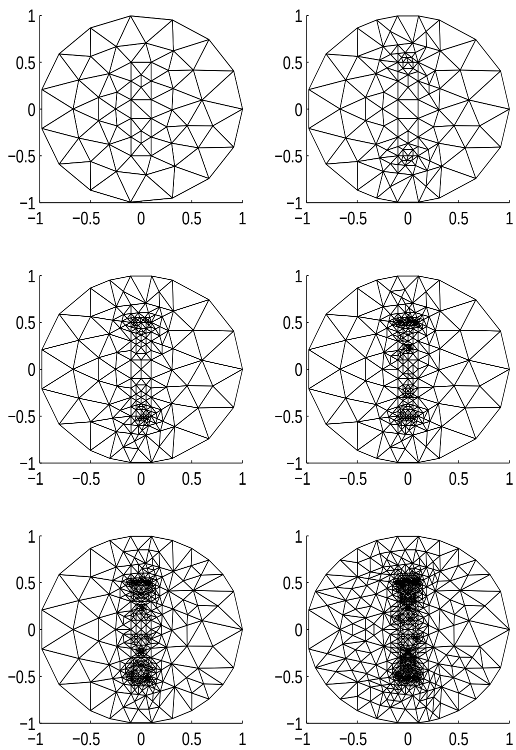

Fig. 6.3. The mesh refinement process for Example 2.

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