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A POSTERIORI ERROR ESTIMATES OF A NON-CONFORMING FINITE ELEMENT METHOD FOR PROBLEMS WITH ARTIFICIAL BOUNDARY CONDITIONS*

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Abstract

An a posteriori error estimator is obtained for a nonconforming finite element approximation of a linear elliptic problem, which is derived from a corresponding unbounded domain problem by applying a nonlocal approximate artificial boundary condition. Our method can be easily extended to obtain a class of a posteriori error estimators for various conforming and nonconforming finite element approximations of problems with different artificial boundary conditions. The reliability and efficiency of our a posteriori error estimator are rigorously proved and are verified by numerical examples.

Mathematics subject classification: 65N15, 65N30, 65N38.
Key words: a posteriori estimate, nonconforming finite element method, artificial boundary conditions

1. Introduction

Many physical and engineering problems such as the electric field and the magnetic field, can be modelled by PDEs on unbounded domains. To efficiently solve such problems by numerical methods, one often introduces proper artificial boundary conditions to translate these problems to bounded domain ones [9,10]. These artificial boundary conditions often have implicit integral forms, which are quite different from those of explicit boundary conditions: Dirichlet, Neumann, or mixed boundary conditions.

Furthermore, when the solutions of the reduced bounded domain problems have some singularities, e.g., singularities arising from re-entrant corners, and singularity of Green's function, adaptive mesh refinement strategy can be very useful to improve the efficiency of the finite element approximations. In this case, a posteriori estimators are often required to identify the regions which need further refinement. There are many different methods for the a posteriori estimation, e.g., the residual estimates [4, 12], the averaging methods [12, 14, 15], etc., however, they are mostly developed for bounded domain problems imposed with explicit boundary conditions.

In this paper, we will develop, for the first time to our knowledge, a reliable and efficient a posteriori estimator for a non-conforming finite element approximation of bounded domain elliptic problems with (at least part of) the boundary conditions given in an implicit integral form. Such problems come naturally from unbounded domain elliptic problems by imposing proper implicit artificial boundary conditions. For simplicity, we consider only a model exterior problem in two dimensions. Our approach, however, also easily applies to more general problems

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defined on unbounded domains, such as problems of the potential of the stray field energy in micromagnetics [11, 13] and the semi-strip field of stationary flow in a channel [10], etc..

The rest of the paper is organized as follows. In Section 2, we illustrate how to apply an artificial boundary method to a unbounded domain model problem to produce an equivalent bounded domain problem with an implicit artificial boundary condition [9–11]. In Section 3, inspired by [4], we introduce an equivalent mixed problem, which serves as a useful tool for the a posteriori error estimation. In Section 4, a non-conforming finite element method for the reduced bounded domain problem is briefly discussed. In Section 5, an a posteriori error estimator for the non-conforming finite element approximation of the model problem is given, and its reliability and efficiency are proved. In Section 6, some numerical examples are given to verify our theoretical results.

2. The Model Problem and the Artificial Boundary Method

We consider a general second-order linear elliptic problem [10]

$$-\text{div}(A\nabla u) + cu = f, \quad \text{in } \Omega, \tag{2.1}$$
$$(A\nabla u)\cdot \mathbf{n} = g, \quad \text{on } \Gamma_N, \tag{2.2}$$
$$u = u_D, \quad \text{on } \Gamma_D, \tag{2.3}$$
$$u - u_\infty \to 0, \quad \text{as } \|x\| \to \infty, \tag{2.4}$$

where $\Omega \subset \mathbf{R}^2$ is a unbounded domain with a Lipschitz boundary $\Gamma = \Gamma_D \bigcup \Gamma_N$ satisfying that the Dirichlet boundary Γ_D is closed, $\Gamma_D \bigcap \Gamma_N = \emptyset$ and the length of the Dirichlet boundary $|\Gamma_D| > 0$ whenever $\Gamma_D \neq \emptyset$, $\mathbf{n}$ is the unit exterior normal to Γ_N, u_∞ is in general a unknown constant and $u_\infty = 0$ when $\Gamma_D = \emptyset$, $c \in L^\infty(\Omega)$ is non-negative and satisfies $c(x) \geq c_0 \geq 0$ for almost all $x \in \Omega$, and the coefficient matrix $A \in L^\infty(\Omega; \mathbf{R}^{2\times 2})$ is symmetric and uniformly positive definite, that is, for some constants $0 < \mu < M < \infty$, there holds

$$\mu\|y\|^2 \leq y \cdot A(x)y \leq M\|y\|^2, \qquad \forall y \in \mathbf{R}^2 \text{ and for a.e. } x \in \Omega.$$

Furthermore, we assume that $\text{supp}(f)$, $\text{supp}(A-I)$, and $\text{supp}(c-c_0)$ are compact, where $\text{supp}(\cdot)$ denotes the support set of a given function, and I is the identity matrix.

For such a problem, if R is sufficiently large so that $\text{supp}(f) \cup \text{supp}(A-I) \cup \text{supp}(c-c_0) \cup \Gamma \subset B(0,R) := \{x : \|x\| < R\}$, then the circle $\Gamma_e := \{x : \|x\| = R\}$ can be taken as an artificial boundary, which divides the unbounded domain Ω into two parts $\Omega_i := \Omega \bigcap B(0,R)$ and $\Omega_e = \{x : \|x\| > R\}$. Artificial boundary conditions can be introduced on $\Gamma_e = \partial B(0,R)$. For simplicity and without loss of generality, we restrict ourselves to the case when $c_0 = 0$, similar artificial boundary conditions for the general case can be found in [10].

Since the solution u to problem (2.1)-(2.4) is harmonic in Ω_e, we have, for $r > R$,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{2.5}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\cos n\phi \, d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\sin n\phi \, d\phi, \tag{2.6}$$

are the Fourier coefficients of $u(R,\theta)$, and we have $u_\infty = a_0/2$.