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ON A MOVING MESH METHOD FOR SOLVING PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS*

Jingtang Ma
School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China
Email: mjt@swufe.edu.cn
Yingjun Jiang[1)]
Department of Mathematics and Scientific Computing, Changsha University of Science and Technology, Changsha 410076, China
Email: jiangyingjun@csust.edu.cn
Kaili Xiang
School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China
Email: xiangkl@swufe.edu.cn

Abstract

This paper develops and analyzes a moving mesh finite difference method for solving partial integro-differential equations. First, the time-dependent mapping of the coordinate transformation is approximated by a piecewise quadratic polynomial in space and a piecewise linear function in time. Then, an efficient method to discretize the memory term of the equation is designed using the moving mesh approach. In each time slice, a simple piecewise constant approximation of the integrand is used, and thus a quadrature is constructed for the memory term. The central finite difference scheme for space and the backward Euler scheme for time are used. The paper proves that the accumulation of the quadrature error is uniformly bounded and that the convergence of the method is second order in space and first order in time. Numerical experiments are carried out to confirm the theoretical predictions.

Mathematics subject classification: 65R20, 65N50, 65M50, 65M20, 65M15, 65M10.
Key words: Partial integro-differential equations, Moving mesh methods, Stability and convergence.

1. Introduction

Moving mesh methods for solving partial differential equations(PDEs) work in the way of moving the mesh points towards the region of large gradient while keeping the number of mesh points fixed during the process. Over the past three decades focuses have been on the development of algorithms for both mesh generation and discretizing the physical PDEs on variable meshes. Blom, Sanz-Serna and Verwer [2] first classify the moving mesh algorithms — BJCN scheme (it can be regarded as a special case of Godunov methods), IEL scheme (implicit-Euler Lagrangian scheme), and RFDM (rezoning finite difference method). After then Tang [16] classifies the moving mesh methods into two general classes: interpolation-free algorithms and interpolation-based algorithms. In interpolation-free algorithms, the mesh equations and

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1) corresponding author.

the original PDEs are solved simultaneously for the physical solution and the mesh, see e.g., [5,8,17]. In the interpolation-based algorithms, the solutions of the mesh equations and physical equations are separate. The algorithms summarized in [2, 10, 15] (see also [17] and references therein) belong to this group. Recently, the analysis on stability and convergence has been brought up to attention. However, little achievement has been made in this direction. Jamet [9] gives a convergence proof of BJCN scheme for the heat equation with moving boundaries, where the mesh evolves uniformly with the moving boundaries. Their proof highly relies on the uniformity of the spatial mesh in each time level, while the technique cannot be used for the variable mesh in each time level. Recently, Ma [13] gives a convergence proof of BJCN scheme on general variable mesh — equidistributing mesh. Mackenzie and Mekwi [14] prove an asymptotic second-order convergence for a conservative IEL scheme which can be regarded as a variant of BJCN scheme. Lipnikov and Shashkov [11] give a rigorous analysis on a rezoning method where the rezoning mesh is generated by minimizing a posteriori error in L_2 norm instead of by an equidistribution principle.

In this paper, we design a stable and second-order scheme for partial integro-differential equations which arise in many applications (e.g., [3, 19] and the references). Although the scheme is more or less motivated by Mackenzie and Mekwi [14], it improves the accuracy to second order using quadratic approximation to the mesh trajectory. In addition, the analysis is given for integro-differential equations. This type of equations can generate singular solutions whose locations are not known a priori. Thus moving mesh methods deserve to solving this type of equations. To discretize the memory term with moving mesh, a piecewise constant polynomial is used to approximate the integrand in each time slice. The accumulation of the quadrature error is proven to be uniformly bounded and thus the stability is derived under several mild assumptions.

In particular, we consider a partial integro-differential equation of the form

$$u_t + au_x - \kappa u_{xx} + \int_0^t k(x,t,s)u(x,s)\,ds = 0, \quad x \in I \equiv [x_L, x_R],\ t \in J \equiv [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in I, \tag{1.2}$$

$$u(x_L,t) = b_L(t), \quad u(x_R,t) = b_R(t), \qquad t \in J. \tag{1.3}$$

where a is a constant advection velocity and κ a constant diffusivity, the integral is called memory term, $k(x,t,s)$ is the kernel function satisfying

$$\max_{x\in I} |k(x,t,s)| \le C|A(t,s)K_\alpha(t-s)|,$$

where A is sufficiently smooth in t and s, and the Hammerstein kernel

$$K_\alpha(t-s) = \begin{cases} (t-s)^{-\alpha}, & 0<\alpha<1, \\ K(t-s), & \text{otherwise}, \end{cases}$$

K is smooth function, $K_\alpha(t-s) = (t-s)^{-\alpha}$ is said to be weakly singular kernel.

The mesh movement is based on the time-dependent mapping

$$x(\cdot,t):\ I_c \equiv [0,1] \to I \equiv [x_L, x_R].$$

Then a function $u(x,t)$ in physical variables is transformed into the function in computational variables

$$u(x,t) = u(x(\xi,t),t).$$