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STABILITY ANALYSIS OF YEE SCHEMES TO PML AND UPML FOR MAXWELL EQUATIONS*

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Abstract

We utilize Fourier methods to analyze the stability of the Yee difference schemes for Bérenger PML (perfectly matched layer) as well as the UPML (uniaxial perfectly matched layer) systems of two-dimensional Maxwell equations. Using a practical spectrum stability concept, we find that the two schemes are spectrum stable under the same conditions for mesh sizes. Besides, we prove that the UPML schemes with the same damping in both directions are stable. Numerical examples are given to confirm the stability analysis for the PML method.

Mathematics subject classification: 65M12.
Key words: Computational electromagnetic, Fourier method, Yee scheme, PML, UPML, Stability.

1. Introduction

A general approach in computational electromagnetic of finding infinite space solutions is to introduce an absorbing boundary condition in the outer lattices boundary to simulate the extension of the lattice to infinity. While an alternative approach is to terminate the outer boundary of the space lattice in an absorbing material medium, the difficulty is that such an absorbing layer is matched only to normally incident plane wave [1].

In 1994, Bérenger published a pioneer paper about the so-called perfectly matched layer (PML) method. By splitting the field, plane waves of arbitrary incidence, polarization, and frequency are matched at the boundary [2]. Since then, the PML has been very popular, and many works from both engineering and mathematical points have been carried out in the fields.

According to Chew and Weedon's observation, the system of the PML medium can be obtained by a complex change of independent variables, which is the famous UPML [5]. Moreover, Sack *et al.* imposed a physical model with perfectly matched medium on an anisotropic parameters without splitting the fields [6]. These two formulations are also mathematically identical, and the fact was proved by Zhao and Cangellaris, provided that the electric and magnetic fields presented in the Chew-Weedon stretched-coordinate formulation are properly defined [9]. Bramble and Pasciak have shown the existence and uniqueness of solutions to the truncated time harmonic PML problem provided that the truncated domain is sufficiently large [10]. Bao

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and Wu have given the convergence analysis in spherical coordinates for a three-dimensional electromagnetic scattering problem and established an explicit error estimate between the solution of the scattering problem and that of the truncated PML problem [11]. Chen and Liu have developed an adaptive PML technique for solving the time harmonic scattering problems. Numerical experiments are included to illustrate the competitive behavior of the proposed adaptive method [12].

Although too many works have been reported about the PML and its successful applications, still many questions are deserved to investigate. The stability of PML is one among them in spite of some known results.

It is well known that the Maxwell's equations are hyperbolic and symmetric; as a result, the initial value problem is well-posed. On the other hand, the Bérenger's PML is not symmetric which may causes instability. However, the damping term is a "good" ingredient to improve stability. The first analysis of the stability is due to the work of Abarbanel and Gottlieb [3]. They studied the split system of equations and proved that if damping parameter $\sigma = 0$ the initial value problem of the system is weakly stable, namely, the L^2-norm of the solutions depend not only on the L^2-norm of the initial data but also on the L^2-norm of their derivatives. They also studied the Yee's scheme to this problem and showed that the numerical solutions to the split TE model with $\sigma = 0$ grows linearly with the time step n, therefore the scheme is unstable. It has been proved that the PML is weakly stable for $\sigma > 0$ as well, for example see [13]. In order to overcome the weakly well-posedness, a lot of authors have designed new modified PMLs, such as, Lions *et al* have imposed a new type of absorbing layer for Maxwell equations and the linearized Euler equations, which is also valid for several classes of first order hyperbolic system, and the associated Cauchy problems are proved well-posed [8]. Besides, Zhao and Cangellaris in [9] proposed a modified PML by restoring the usual operator with a new introduced unknown without splitting fields. Becache and Joly [7] made a thorough investigation for the problem of stability. They proved the weak stability, and the equivalence between the PML and the system in [9]. Moreover, they proved the stability of the initial value problem of the system in [9] for all $\sigma \geq 0$, and the stability of the Yee's scheme to this problem for all $\sigma \geq 0$, too.

It looks like a contradiction that different works generate stable, unstable, or weakly stable results to the same model. In fact, the properties of stability relate to different unknowns in different formulations. For example, the Bérenger's PML is a 4×4 system for the TE mode with unknowns E_x, E_y, H_x and H_y, and the formulation by Zhao and Cangellaris is a 4×4 system with unknowns $E_x, E_y, \tilde{E}_x$ and H_z. Consequently the L^2-norm stability proved in [7] does not apply to the Bérenger's PML directly. Particularly if $\sigma = 0$, the Yee's scheme to the PML is unstable, while the scheme to the formulation by Zhao and Cangellaris is stable.

In this paper, we are interested in the stability property of the Yee's scheme to the PMLs for the case of $\sigma \geq 0$. We will show that the damping parameter $\sigma > 0$ can improve the behavior of stability. The scheme is no longer unstable but stable in a weaker sense which will be called spectrum stability. For the UPML, it may be stable in some cases.

Regarding the stability analysis of PML methods, there are some other related works need to be mentioned. Some more general formulations of PML have been derived by Appelo, Hagstrom and Kreiss, and their stability is analyzed by using Schur criterion in the continuous setting [15]. Ying has considered an exterior initial-boundary value problem of TM mode by truncating the domain with the UPML, and obtained the existence and uniqueness of the weak formulation [14]. Later on, Ying and Fang have analyzed the corresponding FDTD initial-