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A UNIFORM FIRST-ORDER METHOD FOR THE DISCRETE ORDINATE TRANSPORT EQUATION WITH INTERFACES IN X,Y-GEOMETRY*

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Abstract

A uniformly first-order convergent numerical method for the discrete-ordinate transport equation in the rectangle geometry is proposed in this paper. Firstly we approximate the scattering coefficients and source terms by piecewise constants determined by their cell averages. Then for each cell, following the work of De Barros and Larsen [1, 19], the solution at the cell edge is approximated by its average along the edge. As a result, the solution of the system of equations for the cell edge averages in each cell can be obtained analytically. Finally, we piece together the numerical solution with the neighboring cells using the interface conditions. When there is no interface or boundary layer, this method is asymptotic-preserving, which implies that coarse meshes (meshes that do not resolve the mean free path) can be used to obtain good numerical approximations. Moreover, the uniform first-order convergence with respect to the mean free path is shown numerically and the rigorous proof is provided.

Mathematics subject classification: 41A30, 41A60, 65D25.
Key words: Transport equation, Interface, Diffusion limit, Asymptotic preserving, Uniform numerical convergence, X,Y-geometry.

1. Introduction

The transport equation plays an important role in many physical applications, such as neutron transport, radiative transfer, high frequency waves in heterogeneous and random media, semiconductor device simulation and so on. One difficulty about solving this equation numerically is when its mean free path (the average distance a particle travels between two successive interactions with the background media) is small, which requires numerical resolution of the small scale. Historically researchers use the diffusion limit to approximate the solution when the cost is too much to solve the transport equation directly. This small scale is embodied by the introduction of a dimensionless parameter ϵ into the transport equation and the diffusion limit can be obtained when $\epsilon \to 0$.

In this paper we consider the steady state isotropic neutron transport equation with interfaces in the X,Y-geometry. The interface condition to be used is that the density of all directions is continuous at the interface, which often arises in neutron transport equations. There is another kind of interface condition which always arises in radiative transfer equations as an approximation of high frequency waves in random and heterogeneous media, where the energy flux is continuous [2,14,16]. As the interface condition is local, for the density continuous case, we only need to consider the one-dimensional interface analysis which is given in [15].

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The diffusion limit for two-dimensional neutron transport equation with interfaces is derived in this paper. For simplicity, we assume that the incoming particle density at the boundaries is isotropic, in which case no boundary layer exists when $\epsilon \to 0$. A first-order numerical method uniform with respect to ϵ is proposed and investigated here. The most used determined method for this problem has been the discrete ordinate method which is a semi-discretization of the velocity field. Our method is based on the discrete ordinate form and the full discretization is also considered. As in [15], we approximate the coefficients by piecewise constants determined by their cell averages. Then for each cell, following the work of De Barros and Larsen [1,19], the solution at the cell edge is approximated by its average along the edge. The obtained system of equations for the cell edge averages can be solved analytically in each cell and we can get the solution of the whole domain by using the interface conditions to piece together the numerical solution with the neighboring cells. This method is a direct extension of the one-dimensional scheme proposed in [15].

Asymptotic-preserving (AP) schemes have been proved quite successfully for problems with small scales. A scheme with AP properly means it is a good scheme for the original equation, which in the limit as the small parameter goes to zero, becomes an effective scheme for the limit equation [10]. It was proved in [9] for the linear transport equation with boundary conditions that an AP scheme converges *uniformly* with respect to ϵ. This implies we can solve the transport equation without resolving the small scale. For more applications of AP schemes we refer to [5–7] for plasmas and fluids and [3, 11, 12] for hyperbolic systems with stiff relaxations. When there is no interface or boundary layer, the method we propose here is proved to be AP and its uniform first-order accuracy is demonstrated numerically. The rigorous convergence proof relies heavily on the eigenfunction expansion of the constant coefficient, one-dimensional discrete-ordinate equations, but the idea is quite similar to the proof of uniform second-order convergence for one-dimensional case in [15]. AP means that at the interior of the materials where the solution varies slowly (no matter whether ϵ is big or small), we do not need to resolve ϵ to get accurate approximations, though the equation itself requires under-resolving. When using coarse meshes, most methods can not provide satisfactory results for problems with boundary layers even if they are AP inside. The method proposed in [15] seems the first effort that is uniformly convergent valid up to the boundary. It will be demonstrated numerically that our two-dimensional method is not valid at the layers, but this can be improved by resolving ϵ locally at the boundaries or interfaces.

Similar ideas can be found in [1, 17, 19] but with auxiliary equations and in an iterative way. They only discussed piecewise constant case and the required storage is much more than that of our approach. Problems with coefficients depending on space are investigated in this paper and we also present its AP property and uniform accuracy with respect to ϵ.

The arrangement of this paper is as follows. In section 2, we introduce the two-dimensional neutron transport equation and its discrete ordinate form, and derive their diffusion limit with interfaces. In section 3, the scheme is given and its AP property is proved in section 4. Several numerical examples are presented in section 5 to test the AP property, and the uniform accuracy is discussed. Finally we make some conclusions in section 6.

2. Neutron Transport Equation in Two-Dimensions

The steady state, isotropic, neutron transport equation in the X,Y-geometry reads as: for $\mathbf{z} \in \Omega \subset \mathbb{R}^2$, $\mu \in S = \{\mathbf{u} \in \mathbb{R}^2 : |\mathbf{u}| = 1\}$,