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# CONSTRUCTION OF GEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS FOR LEVEL SETS *

Chong Chen
*LSEC, ICMSEC, Academy of Mathematics and Systems Science,*
*Chinese Academy of Sciences; Graduate University of Chinese Academy of Sciences, Beijing 100190, China*
*Email: chench@lsec.cc.ac.cn*
Guoliang Xu
*LSEC, ICMSEC, Academy of Mathematics and Systems Science,*
*Chinese Academy of Sciences, Beijing 100190, China*
*Email: xuguo@lsec.cc.ac.cn*

**Abstract**

Geometric partial differential equations of level-set form are usually constructed by a variational method using either Dirac delta function or co-area formula in the energy functional to be minimized. However, the equations derived by these two approaches are not consistent. In this paper, we present a third approach for constructing the level-set form equations. By representing various differential geometry quantities and differential geometry operators in terms of the implicit surface, we are able to reformulate three classes of parametric geometric partial differential equations (second-order, fourth-order and sixth-order) into the level-set forms. The reformulation of the equations is generic and simple, and the resulting equations are consistent with their parametric form counterparts. We further prove that the equations derived using co-area formula are also consistent with the parametric forms. However, these equations are of much complicated forms than these given by the equations we derived.

*Mathematics subject classification:* 65D17.
*Key words:* Geometric partial differential equations, Level set, Differential geometry operators.

## 1. Introduction

In many scientific research and application areas of surfaces, such as geometry design (see, e.g., [1–4]), shape deformation (see, e.g., [5–9]), surface reconstruction (see, e.g., [10, 11]), surface restoration (see, e.g., [12, 13]) and image processing (see, e.g., [14, 15]), geometric partial differential equations (PDEs), which govern the motion of surfaces, have played a very important role. Using geometric PDEs, a number of efficient and effective numerical methods have been obtained, usually called geometric PDE method. The basic theory and numerical methods concerned geometric PDEs can be found in many references. We suggest the interested readers to refer [3, 14, 16, 17].

Depending on the nature of the problems to be solved, geometric PDEs are represented as either parametric form or level-set form. By virtue of surface variation techniques, a vast geometric PDEs have been successfully derived by minimizing certain energy functionals defined

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on parametric surfaces (see [3] and literatures therein). However, less efforts have been made for implicit surfaces. Currently, two variational methods have been developed. The first one utilizes the Dirac delta function in the energy functional to convert the energy functional defined on a surface to an energy functional defined on a 3D volume. Then the first order variational calculus is conducted over the 3D domain and an Euler-Lagrange equation is finally derived. The second technique employs the co-area formula to convert the surface energy to the volume energy (see [13, 16]) and from there the Euler-Lagrange equation is obtained. By the second approach, the second-order, the fourth-order and sixth-order geometric PDEs in the general form have been constructed (see [16]). It has been behind that the two approaches result in the same equations and they are all equivalent to the parametric form ones for the same energy functional. However, these claims are not theoretically proved. Our recent study shows that the equations constructed via these two approaches are very different.

We insist that the right equation of the level-set form should coincide with the parametric form if the starting energy to be minimized is the same. This motivates strongly the current research. The aim of this paper is to construct geometric PDEs in the level-set form such that the constructed equations are equivalent to their parametric form counterpart for the same energy functional. Our strategy is that converting directly these geometric PDEs from parametric form to implicit form, so that the constructed geometric PDEs in level-set forms are equal to those in the parametric forms. In order to accomplish this task, we need to convert all the required geometric quantities and various differential geometry operators, that are used to describe geometric PDEs and previously defined for the parametric surface, from the parametric formulations to the level-set ones.

In the classical differential geometry, various geometric quantities on parametric surface have been introduced and widely used. For instance, various curvatures and geometric operators (see Section 2) are well understood (see [18–20]). For implicit surface, some of these geometric entries, such as Gaussian curvature, mean curvature and principal curvatures, have been given in the literatures (see [20–24]). But some of the others, such as the principal directions, the second and third tangent operators, and Giaquinta-Hildebrandt operator, have not been defined for the implicit surfaces, to the best of our knowledge. These operators are necessary for representing geometric PDEs in the implicit form.

Our contributions in this paper include: (i) Generalize the parametric form differential operators to level-set surface; (ii) Convert geometric PDEs in parametric forms directly into level-set forms; and (iii) Prove the equivalent relationship between the equations derived using co-area formula and the equations obtained via our approach.

The rest of the paper is organized as follows. In Section 2, we first review some results on parametric form differential geometry, and then in Section 3 we represent some useful differential quantities and differential operators in the level-set form. In Section 4 we reformulate the parametric form geometric PDEs in the level-set form. The equivalency of our PDEs and the ones using co-area approach is discussed in Section 5. Section 6 concludes the paper.

## 2. Mathematical Preliminaries

This section introduces the notations and fundamental mathematics used in the following context. We first give in Subsection 2.1 some of the main geometric notions and various results of differential operators defined on parametric surfaces. Subsection 2.2 presents some geometric partial differential equations in parametric form which are frequently used in computational