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NODAL $\mathcal{O}(h^4)$-SUPERCONVERGENCE IN 3D BY AVERAGING PIECEWISE LINEAR, BILINEAR, AND TRILINEAR FE APPROXIMATIONS*

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Abstract

We construct and analyse a nodal $\mathcal{O}(h^4)$-superconvergent FE scheme for approximating the Poisson equation with homogeneous boundary conditions in three-dimensional domains by means of piecewise trilinear functions. The scheme is based on averaging the equations that arise from FE approximations on uniform cubic, tetrahedral, and prismatic partitions. This approach presents a three-dimensional generalization of a two-dimensional averaging of linear and bilinear elements which also exhibits nodal $\mathcal{O}(h^4)$-superconvergence (ultraconvergence). The obtained superconvergence result is illustrated by two numerical examples.

Mathematics subject classification: 65N30.
Key words: Higher order error estimates, Tetrahedral and prismatic elements, Superconvergence, Averaging operators.

1. Introduction

We consider the Poisson equation with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded rectangular domain and that the right-hand side function $f \in C^4(\overline{\Omega})$.

The weak form of problem (1.1) reads: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the scalar products in both $L_2(\Omega)$ and $(L_2(\Omega))^3$.

In [15], Schatz referred about the nodal $\mathcal{O}(h^4)$-superconvergence of quadratic elements on uniform tetrahedral partitions (i.e., for each internal edge e the patch of tetrahedra sharing e

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is a point symmetric set with respect to the midpoint of e). This result is extended by Schatz, Sloan, and Wahlbin [16] to locally symmetric meshes. Since each uniform tetrahedralization is locally point-symmetric with respect to the midpoints of edges, the $\mathcal{O}(h^4)$-superconvergence of quadratic tetrahedral elements holds at these midpoints as well.

Linear triangular elements also exhibit nodal $\mathcal{O}(h^4)$-superconvergence (ultraconvergence) on uniform triangulations consisting solely of equilateral triangles. This result was obtained by Lin and Wang in [14] (see also [2]). It is based on the fact that the corresponding stiffness FE matrix is the same as the matrix associated to the standard 7-point finite difference scheme, which is $\mathcal{O}(h^4)$-accurate. However, this result cannot be extended to three-dimensional space, since the regular tetrahedron is not a space-filler (see, e.g., [4, 11]).

The study of superconvergence by a computer-based approach developed by Babuška et al. [1] requires to examine harmonic polynomials in the plane. Note that the dimension of the space of harmonic polynomials of degree $k \in \{1, 2, \dots\}$ in two variables is only 2, whereas the dimension of such a space in three variables is $2k+1$. This makes superconvergence analysis for $d = 3$ much more difficult (see, e.g., [17]) than for $d = 2$. The likelihood of $2k + 1$ polynomial graphs passing through a common point is much smaller than the probability of two intersecting polynomial graphs.

A suitable averaging of gradients of FE solutions leads to superconvergence, see, e.g., [5,6]. In this paper we show that an averaging of stiffness matrices of several kind of elements exhibits also a superconvergence. In particular, here we will present an averaging of linear algebraic equations arising from FE approximations of problem (1.2) on uniform partitions of $\overline{\Omega}$ into cubes, tetrahedra, and triangular prisms, respectively. The method is an extension of the nodal $\mathcal{O}(h^4)$-superconvergence result for the Poisson equation in two-dimensional domains, where the stiffness matrices corresponding to linear and bilinear elements are appropriately averaged [12,13] to obtain the matrix associated to the standard 9-point finite difference scheme. To the authors' knowledge, extension of this result to the three-dimensional case has not yet been studied. Note that the size (and also the band-width) of the resulting matrix will be the same as for the stiffness matrix corresponding to trilinear finite elements, which produces only $\mathcal{O}(h^2)$-accuracy in the maximum norm at nodes.

2. Construction of the Averaged FE Scheme

2.1. Preliminaries

Assume that $\mathcal{T}_h$ is a uniform face-to-face partition of the domain $\overline{\Omega}$ into cubes. We denote the set of interior nodes of $\mathcal{T}_h$ by $\mathcal{N}_h = \{z_i\}_{i=1}^N$, where $N = N(h)$ and h is the length of any edge.

In order to introduce the relevant FD and FE schemes, we shall use the compact notation from [9]. To this end, the nodes in the FD stencil (see Figure 2.1) are divided into three separate groups (midpoints of faces, vertices, and midpoints of edges) and the following conventional summations

$$\begin{aligned}
\Diamond U_0 &= U_1 + U_2 + U_3 + U_4 + U_{13} + U_{14}, \\
\bigcirc U_0 &= U_{19} + U_{20} + U_{21} + U_{22} + U_{23} + U_{24} + U_{25} + U_{26}, \\
\square U_0 &= U_5 + U_6 + U_7 + U_8 + U_9 + U_{10} + U_{11} + U_{12} + U_{15} + U_{16} + U_{17} + U_{18},
\end{aligned}$$

are used, where the value U_0 corresponds to the (central) vertex z_i and $U_1, \dots, U_{26}$ stand for