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NUMERICAL IMPLEMENTATION FOR A 2-D THERMAL INHOMOGENEITY THROUGH THE DYNAMICAL PROBE METHOD *

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Abstract

In this paper, we present the theory and numerical implementation for a 2-D thermal inhomogeneity through the dynamical probe method. The main idea of the dynamical probe method is to construct an indicator function associated with some probe such that when the probe touch the boundary of the inclusion the indicator function will blow up. From this property, we can get the shape of the inclusion. We will give the numerical reconstruction algorithm to identify the inclusion from the simulated Neumann-to-Dirichlet map.

Mathematics subject classification: 35R30.
Key words: Heat equation, Dynamical probe method, Neumann-to-Dirichlet map.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$. We consider a heat conductor Ω with an inclusion D such that $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ is connected and the boundary ∂D of D is of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$. Let the heat conductivity $\gamma(x)$ in Ω be given as follows:

$$\gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega \setminus \overline{D} \\ k & \text{for } x \in D \end{cases}$$

with a positive constant k which is not 1. That is, by using the characteristic function χ_D of D, $\gamma(x)$ is given as $\gamma(x) = 1 + (k-1)\chi_D$.

Here Ω could be a cross section of a heat conductive bar which contains an unknown inclusion with a uniform cross section D. We are concerned with a thermographic nondestructive testing to identify D. This testing is to identify D from the measurements which apply heat flux (sometime called thermal load) to $\partial\Omega$ many times and measure the corresponding temperature on $\partial\Omega$. For more details, the readers can refer to [15], [16] and the references therein. In this paper, we will provide both theoretical and numerical schemes for this testing.

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First of all, we will give some notations which will be used throughout this paper. For a set B, $B \times (T_1, T_2)$ and $B \times (0, T)$ are denoted by $B_{(T_1,T_2)}$ and B_T, respectively. Also, for $p, q \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ or $p = \frac{1}{2}$, $H^p(\Omega), H^p(\partial\Omega)$ and $H^{p,q}(\Omega_T)$ denote the usual Sobolev spaces, where p and q in $H^{p,q}(\Omega_T)$ denote the regularity with respect to x and t, respectively (cf. [12]). Further, for an open set $U \subset \mathbb{R}^3$ with Lipschitz boundary ∂U and $p, q \in \mathbb{Z}_+$, $H^{p,q}(U)$ is defined similar to $H^{p,q}(\Omega_T)$. That is $g \in H^{p,q}(U)$ if and only if there exists $\mathbf{g} \in H^{p,q}(\mathbb{R}^3)$ with $g = \mathbf{g}$ in U. The norm $||g||_{H^{p,q}(U)}$ of g defined by

$$||g||_{H^{p,q}(U)} := \inf \left\{ ||\mathbf{g}||_{H^{p,q}(\mathbb{R}^3)}; \mathbf{g} \in H^{p,q}(\mathbb{R}^3) \text{ and } \mathbf{g}|_U = g \right\}.$$

Moreover, a function $f(x,t)$ is in $L^2((0,T); X)$ if $f(\cdot, t) \in X$ for almost all $t \in (0,T)$ and

$$||f||^2_{L^2((0,T);X)} = \int_0^T ||f(\cdot,t)||^2_X dt < \infty.$$

The forward problem for the thermographic nondestructive testing is to find a unique weak solution $u = u(f) \in H^{1,0}(\Omega_T)$ which satisfies

$$\begin{cases} \mathcal{P}_D u(x,t) := \partial_t u(x,t) - \operatorname{div}_x(\gamma(x)\nabla_x u(x,t)) = 0 \quad \text{in } \Omega_T \\ \partial_\nu u(x,t) = f(x,t) \quad \text{in } \partial\Omega_T, \quad u(x,0) = 0 \quad \text{for } x \in \Omega \end{cases} \tag{1.1}$$

for a given $f \in L^2((0,T); (H^{1/2}(\partial\Omega))^*)$. Namely, by assuming the initial temperature of a heat conductive medium Ω is 0, determine the temperature $u = u(f)$ induced in Ω_T after applying the heat flux f on $\partial\Omega_T$.

By a weak solution $u = u(f) \in H^{1,0}(\Omega_T)$ of Problem (1.1), we mean a function $u = u(f)$ which satisfies

$$\int_{\Omega_T} (-u\partial_t\varphi + \gamma(x)\nabla_x u \cdot \nabla_x \varphi) dx dt = \int_{\partial\Omega_T} f\varphi|_{\partial\Omega_T} d\sigma dt$$

for all

$$\varphi \in W(\Omega_T) := \left\{ v \in H^{1,0}(\Omega_T); \partial_t v \in L^2((0,T); (H^1(\Omega))^*) \right\}$$

with $\varphi(x,T) = 0$ for all $x \in \Omega$.

It is well known that the boundary value problem (1.1) is well posed (see [17]). That is there exists a unique solution $u = u(f) \in H^{1,0}(\Omega_T)$ to (1.1) and $u(f)$ depends continuously on $f \in L^2((0,T); (H^{1/2}(\partial\Omega))^*)$. Based on this, we define the *Neumann-to-Dirichlet map* Λ_D : $L^2((0,T); (H^{1/2}(\partial\Omega))^*) \to L^2((0,T); H^{1/2}(\partial\Omega))$ by $\Lambda_D(f) = u(f)|_{\partial\Omega_T}$.

Now, we take the Neumann-to-Dirichlet map Λ_D as measured data for our nondestructive testing. Then, our inverse problem is to reconstruct the unknown inclusion D from Λ_D.

In [3], authors gave a reconstruction procedure for one space dimensional case. It is an analogue of the probe method which was introduced by Ikehata [7] to identify the shape of unknown inclusion in a stationary heat conductive medium. They gave a theory on how to adapt the probe method for the stationary heat conductive case and provided a reconstruction scheme identifying an inclusion which can depend on time for one space dimensional case. Below, we will refer this kind of dynamical version of the probe method by *dynamical probe method*. Further, Isakov, Kim, and Nakamura [8] extended this argument and established the foundation for dynamical probe method.

Isakov, Kim, and Nakamura gave the proof of probe method for the three dimensional case in [8]. As the proof is quite different for the two space dimensional case, in this work we will