UNIFORM SUPERCONVERGENCE OF A FINITE ELEMENT METHOD WITH EDGE STABILIZATION FOR CONVECTION-DIFFUSION PROBLEMS[*]

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Abstract

In the present paper the edge stabilization technique is applied to a convection-diffusion problem with exponential boundary layers on the unit square, using a Shishkin mesh with bilinear finite elements in the layer regions and linear elements on the coarse part of the mesh. An error bound is proved for $\left\| \pi u - u^h \right\|_E$, where πu is some interpolant of the solution u and u^h the discrete solution. This supercloseness result implies an optimal error estimate with respect to the L_2 norm and opens the door to the application of postprocessing for improving the discrete solution.

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1. Introduction

Consider the model convection-diffusion problem

$$-\varepsilon \Delta u - \boldsymbol{b} \cdot \nabla u + cu = f \quad \text{in } \Omega = (0,1)^2, \quad u = 0 \quad \text{on } \Gamma = \partial\Omega \tag{1.1}$$

with a small perturbation parameter $0 < \varepsilon \ll 1$ and $b_i \geq \beta_i > 0$ on $\bar{\Omega}$, $i = 1, 2$, with constants β_i. Furthermore, let

$$c + \frac{1}{2} \operatorname{div} \boldsymbol{b} \geq \gamma > 0 \quad \text{on } \bar{\Omega}. \tag{1.2}$$

This ensures the existence of a weak solution in $H_0^1(\Omega) \cap H^2(\Omega)$. The presence of the parameter ε causes the formation of regular layers at the outflow boundary at $x = 0$ and $y = 0$.

It is well known that the Galerkin finite element method applied to (1.1) with linear or bilinear finite elements leads on standard meshes to wild non-physical oscillations in the discrete solution.

The need of avoiding these oscillations and to resolve layer structures results in the use of layer-adapted meshes which are highly anisotropic and non-uniform. On the probably simplest layer-adapted mesh, the Shishkin mesh (see Section 2), standard Galerkin with $\mathcal{O}\left(N^2\right)$ mesh points was shown to converge uniformly by Stynes and O'Riordan [18] ten years ago:

$$\left\| \left| u - u^h \right| \right\|_\varepsilon \leq CN^{-1} \ln N. \tag{1.3}$$

where here and throughout C is a constant independent of ε and $\|\|v\|\|_\varepsilon := \varepsilon^{1/2}|v|_1 + \gamma\|v\|_0$ is the ε-weighted H^1 norm. The fine mesh in the layer region induces some stability, but the computed solution still exhibits mild oscillations (see the numerical experiments in [12]). Moreover, the stiffness matrix of the generated discrete problem has eigenvalues with large imaginary parts, and consequently standard iterative methods do not solve the discrete systems efficiently. Therefore, additional stabilization seems to be necessary even when layer-adapted meshes are used.

Stynes and Tobiska [19] analyzed stabilization by the streamline-diffusion finite element method (SDFEM) on the coarse part of the Shishkin mesh. But the relatively popular SD-FEM has some remarkable disadvantages: applied to systems of differential equations it causes non-physical couplings of unknowns [14], while for optimal control problems difficulties in the adjoint equation arise [2]. Consequently, recently new stabilization techniques as local projection schemes, the variational multiscale methods and edge stabilization appeared (for a survey, see [16]).

In the present paper we shall consider instead an edge stabilization technique, which is also called continuous interior penalty method (CIP). It was introduced and analyzed by Burman and Hansbo [4,5]. On a shape-regular, locally uniform mesh they proved for linear elements

$$\|\|u - u^h\|\|_E \leq C\big(\varepsilon^{1/2} + h^{1/2}\big)h|u|_2.$$

In the convection-dominated case this estimate is of little worth because $|u|_2 \to \infty$ for $\varepsilon \to 0$. However, the local estimates of [6] show that edge stabilization is fine away from the layers.

It is our aim to combine the Galerkin finite element method with bilinears on the fine part of a Shishkin mesh with edge stabilization for linear elements on the coarse part of the mesh. Numerical experiments have shown that bilinear elements should be preferred to linear elements in layer regions because they yield higher-order accuracy; see [13]. This is due to supercloseness properties of bilinear elements, cf. [10,20]. This seems to contradict "normal" mesh design where triangles should be used at the boundary of the domain. However, in the singularly perturbed case the situation is different because of the highly anisotropic behaviour of the solution near boudaries. First rectangular boundary-layer meshes are constructed and second the remaining domain is triangulated; see also [15].

Similar to streamline-diffusion it is impossible to get a better estimate then (1.3) for the error in the energy norm. However, for the difference of the numerical solution and a certain interpolant of the exact solution we shall prove in a norm stronger than the ε-weighted norm

$$\|\|\pi u - u^h\|\|_E \leq C\big(\varepsilon^{1/2}N^{-1} + N^{-3/2}\big).$$

This is a supercloseness result and allows for the application of postprocessing techniques giving a new approximate solution Pu^h whose error is significantly smaller than that of u^h.

2. Derivative Bounds and Mesh Construction

For the construction of properly adapted meshes and for the analysis of the resulting method it is essential to have precise knowledge of the behaviour of the solution and its derivatives. As in [10] we shall assume the solution of (1.1) admits the representation

$$u = S + E_1 + E_2 + E_{12}, \tag{2.1a}$$

with

$$\left|\partial_x^i \partial_y^j S(x,y)\right| \leq C, \tag{2.1b}$$

$$\left|\partial_x^i \partial_y^j E_1(x,y)\right| \leq C\varepsilon^{-i} e^{-\beta_1 x/\varepsilon}, \tag{2.1c}$$

$$\left|\partial_x^i \partial_y^j E_2(x,y)\right| \leq C\varepsilon^{-j} e^{-\beta_2 y/\varepsilon} \tag{2.1d}$$

$$\left|\partial_x^i \partial_y^j E_{12}(x,y)\right| \leq C\varepsilon^{-(i+j)} e^{-(\beta_1 x + \beta_2 y)/\varepsilon} \tag{2.1e}$$

for $0 \leq i + j \leq 3$ and all $(x,y) \in \bar{\Omega}$. Conditions that guarantee the existence of such a decomposition can be derived using the technique from [12].

In order to resolve the boundary layers in u we shall use a so called Shishkin mesh [17]. This is constructed as follows. Define the mesh transition points

$$\lambda_i := \min\left\{\frac{1}{2}, \sigma \frac{\varepsilon}{\beta_i} \ln N\right\}, \quad i = 1, 2,$$

with a parameter $\sigma > 0$ that will be defined later. We assume

$$\lambda_i = \frac{\sigma\varepsilon}{\beta_i} \ln N \leq \frac{1}{2} \tag{2.2}$$

as otherwise N is exponentially large compared to $1/\varepsilon$ and the method can be analysed in a conventional manner.
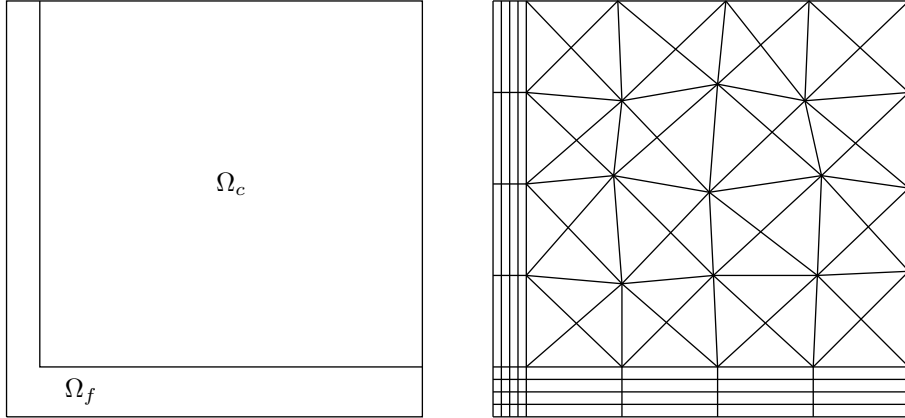

Fig. 2.1. Dissection and triangulation $\mathcal{T}^N$ of Ω

We dissect Ω into two subdomains $\Omega_c := (\lambda_1, 1) \times (\lambda_2, 1)$ and $\Omega_f := \Omega \setminus \bar{\Omega}_c$, see Fig 2.1. The set of mesh points (x_i, y_j) of the Shishkin mesh on Ω_f is given by

$$x_i = \begin{cases} 2i\lambda_1/N, & \text{for } i = 0, \ldots, \frac{N}{2} \\ 1 - 2(N-i)(1-\lambda_1)/N, & \text{for } i = \frac{N}{2}, \ldots, N \end{cases}$$

and

$$y_j = \begin{cases} 2j\lambda_2/N, & \text{for } j = 0, \ldots, \frac{N}{2} \\ 1 - 2(N-j)(1-\lambda_2)/N, & \text{for } j = \frac{N}{2}, \ldots, N. \end{cases}$$

Drawing lines parallel to the x- and y-axes through these mesh points we obtain a triangulation of the domain Ω_f into rectangles. On Ω_c we shall use an arbitrary quasi-uniform triangulation of mesh width $h = \mathcal{O}(N^{-1})$ into triangles, see Fig. 2.1. The combined triangulation is $\mathcal{T}^N$ with the obvious subsets $\mathcal{T}_c^N$ and $\mathcal{T}_f^N$. Elements of the triangulation will be denoted by τ and its edges by e with $\mathcal{E}_c^N$ being the set of all interior edges of $\mathcal{T}_c^N$.

For the formulation and analysis of our method define

$$\Omega_c^* := (x_{N/2-1}, 1) \times (y_{N/2-1}, 1) \quad \text{and} \quad \Omega_t := \Omega_c^* \setminus \bar{\Omega}_c.$$

Thus Ω_c^* is Ω_c extended by a ply of rectangles into the layer region Ω_f, while the transition region Ω_t is the union of these elements.

3. Discretization

We start from a variational formulation of (1.1). The boundary conditions on the inflow part Γ_c^{*-} of Ω_c^* will be imposed weakly. Note that $\Gamma_c^{*-} = \{(x,y) \in \Gamma_c^{*-} : -\boldsymbol{b} \cdot n < 0\}$, i.e., the north and east sides of Ω_c^*. Let

$$V := \left\{ v \in H^1(\Omega) : v|_{\Gamma \setminus \Gamma_c^{*-}} \equiv 0 \right\}.$$

Then a weak formulation of (1.1) is: Find $u \in V$ such that

$$\tilde{a}(u, v) = (f, v) \quad \forall v \in V \tag{3.1a}$$

with

$$\tilde{a}(u, v) := a(u, v) + a_{bc}(u, v), \quad a(u, v) := \varepsilon \left(\nabla u, \nabla v \right) - (\boldsymbol{b} \cdot \nabla u - cu, v) \tag{3.1b}$$

and

$$a_{bc}(u, v) := \int_{\Gamma_c^{*-}} \boldsymbol{b} \cdot n u v - \varepsilon \int_{\Gamma_c^{*-}} \left(\partial_n u v + u \partial_n v \right) + \gamma_{bc} \sum_{e \subset \Gamma_c^{*-}} \frac{\varepsilon}{h_e} \int_e u v, \tag{3.1c}$$

where ∂_n is the outward-normal derivative of Ω and h_e the length of the edge e, while $(\cdot, \cdot)_D$ denotes the standard scalar product in $L_2(D)$. If $D = \Omega$ we drop Ω from the notation. The parameter $\gamma_{bc} > 0$ penalizes the violation of the boundary conditions. Note, if (1.1) possesses a solution $u \in H^2(\Omega)$ it solves (3.1a) too. On V let us define the standard energy norm

$$\|v\|_\varepsilon^2 := \varepsilon \|\nabla v\|_0^2 + \gamma \|v\|_0^2$$

and

$$\|v\|^2 := \|v\|_\varepsilon^2 + \frac{1}{2} \int_{\Gamma_c^{*-}} \boldsymbol{b} \cdot n v^2 + \gamma_{bc} \sum_{e \subset \Gamma_c^{*-}} \frac{\varepsilon}{h_e} \int_e v^2.$$

This is a norm because $\boldsymbol{b} \cdot n > 0$ on Γ_c^{*-}. Furthermore

$$- (\boldsymbol{b} \cdot \nabla v, v) = \frac{1}{2} \left(\operatorname{div} \boldsymbol{b}, v^2 \right) - \frac{1}{2} \int_{\Gamma_c^{*-}} \boldsymbol{b} \cdot n v^2 \quad \forall \, v \in V.$$

Therefore

$$\tilde{a}(v,v) \geq \||v\||^2 \quad \forall \, v \in V,$$

i.e., $\tilde{a}(\cdot,\cdot)$ is coercive on V with respect to the norm $\||\cdot\||$.

Introducing the FE space

$$V^h := \left\{ v \in V : v|_\tau \in P_1(\tau) \ \ \forall \, \tau \in \mathcal{T}_c^N \ \text{ and } \ v|_\tau \in Q_1(\tau) \ \ \forall \, \tau \in \mathcal{T}_f^N \right\},$$

an immediate discretization of (3.1a) is: Find $u^h \in V^h$ such that

$$a(u^h, v^h) = \left(f, v^h\right) \quad \forall v^h \in V^h.$$

In order to introduce additional stability we penalize jumps in the gradient of u^h across interelement boundaries on Ω_c. Let

$$J(w,v) := \gamma_J h^2 \sum_{e \in \mathcal{E}_c^N} \int_e [\nabla w] \cdot [\nabla v],$$

where $[v]$ denotes the jump of v across the edge e. Note, if $u \in H^2(\Omega)$ then $J(u,v) = 0$ for all $v \in V$. Therefore the term $J(u,v)$ can be added to the bilinear form $a(\cdot,\cdot)$ without affecting the Galerkin orthogonality property. Moreover, if $\gamma_J \geq 0$, then

$$J(v,v) \geq 0 \quad \forall \, v \in V,$$

which yields improved stability.

Our final discretization is: Find $u^h \in V^h$ such that

$$a_E\left(u^h, v^h\right) := \tilde{a}\left(u^h, v^h\right) + J\left(u^h, v^h\right) = (f, v^h) \quad \forall \, v^h \in V^h. \tag{3.2}$$

This bilinear form is coercive with

$$\begin{aligned} a_E\left(v^h, v^h\right) &= \tilde{a}\left(v^h, v^h\right) + J\left(v^h, v^h\right) \\ &\geq \||v^h\||^2 + J\left(v^h, v^h\right) =: \||v^h\||_E^2 \quad \forall \, v^h \in V^h. \end{aligned} \tag{3.3}$$

The discretization is consistent if $u \in H^2(\Omega)$:

$$a_E\left(u - u^h, v^h\right) = 0 \quad \forall \, v^h \in V^h. \tag{3.4}$$

4. Analysis of the Method

In the error analysis we shall use a special projection operator that combines the L_2 projection on the coarse mesh region Ω_c with the standard nodal interpolant in the layer region. For $w \in L_2(\Omega_c)$ let $\pi_2 w$ denote its L_2 projection onto

$$V_c^h := \left\{ v \in H^1(\Omega_c) : v|_\tau \in P_1(\tau) \ \ \forall \, \tau \in \mathcal{T}_c^N \right\},$$

i.e.,

$$\pi_2 : w \mapsto \pi_2 w \in V_c^h \quad \text{with} \quad \left(\pi_2 w - w, v^h\right) = 0 \ \ \forall \, v_h \in V_c^h.$$

Given a function $w \in V$, our special projection π is defined by

$$\pi : w \mapsto \pi w \in V^h \quad \text{with} \quad (\pi w)(x_i, y_j) = \begin{cases} (\pi_2 w)(x_i, y_j) & \text{if } i, j \geq N/2, \\ w(x_i, y_j) & \text{otherwise.} \end{cases}$$

Let w^I denote that nodal interpolant of w that is piecewise bilinear on Ω_f and linear on Ω_c. Then $\pi w = w^I$ on $\Omega \setminus \Omega_c^*$ and $\pi w = \pi_2 w$ on Ω_c, while on Ω_t we have a bilinear blending of w^I and $\pi_2 w$.

The error analysis follows standard paths and starts from the coercivity and the Galerkin orthogonality of the method. Let $\chi := \pi u - u^h$ and $\eta := \pi u - u$. Then

$$\|\chi\|_E^2 \leq a_E(\chi, \chi) = a_E(\eta, \chi) = a(\eta, \chi) + a_{bc}(\eta, \chi) + J(\eta, \chi).$$

Denoting by $a.(\cdot, \cdot)_D$ that the integrations and summations in the bilinear form are restricted to D, we can split the right-hand side of the last inequality and obtain

$$\begin{aligned} \|\chi\|_E^2 \leq{} & a\left(u^I - u, \chi\right)_{\Omega_f} + a\left(\pi u - u^I, \chi\right)_{\Omega_t} + a_{bc}(\pi u - u, \chi)_{\Gamma_c^* - \cap \partial \Omega_t} \\ & + a\left(\pi_2 u - u, \chi\right)_{\Omega_c} + a_{bc}(\pi_2 u - u, \chi)_{\Gamma_c^* - \cap \partial \Omega_c} + J(\pi_2 u - u, \chi). \end{aligned} \tag{4.1}$$

The first term can be bounded using results from [10, 20] (see also [11]):

$$a\left(u^I - u, \chi\right)_{\Omega_f} \leq C N^{-2} \ln^2 N \, \|\chi\|_\varepsilon \quad \text{for arbitrary } \sigma > 0. \tag{4.2}$$

For the remaining terms we need estimates for the L_2-projection error on Ω_c. These will be provided in the next section after which we return to the analysis of the approximation error.

4.1. Projection-error estimates

For the nodal interpolant on a Shishkin mesh with $\sigma \geq 2$ we have the bounds

$$\left\|u - u^I\right\|_\varepsilon \leq C N^{-1} \ln N \quad \text{and} \quad \left\|u - u^I\right\|_\infty \leq \begin{cases} C N^{-2} \ln^2 N & \text{on } \bar{\Omega}_f, \\ C N^{-2} & \text{on } \bar{\Omega}_c, \end{cases} \tag{4.3}$$

see [8, 11]. These bounds are sharp.

Lemma 4.1. *The L_2-projection on Ω_c possesses the following stability properties:*

$$\|\pi_2 w\|_{s, \Omega_c} \leq \|w\|_{s, \Omega_c} \quad \forall w \in H^s(\Omega_c), \ s = 0, 1,$$
$$\|\pi_2 w\|_{\infty, \Omega_c} \leq \|w\|_{\infty, \Omega_c} \quad \forall w \in L_\infty(\Omega_c).$$

Proof. The triangulation on Ω_c is shape-regular and locally quasi uniform. Therefore the results from [7] apply. $\square$

Lemma 4.2. *There exists a constant C such that for any $\tau \in \mathcal{T}_c^N$ the following trace inequality holds*

$$\|w\|_{0, \partial \tau}^2 \leq C\left(h^{-1} \|v\|_{0, \tau}^2 + h \|v\|_{1, \tau}^2\right) \quad \forall v \in H^1(\tau)$$

Proof. See [3]. $\square$

Theorem 4.1. *Let u be the solution of* (1.1). *Then on a Shishkin mesh with $\sigma \geq 2$*

$$
\begin{array}{lll}
(i) & & \|u - \pi_2 u\|_{0,\Omega_c} \leq Ch^2, \\[4pt]
(ii) & & \varepsilon^{1/2} \|\nabla (u - \pi_2 u)\|_{0,\Omega_c} \leq C\left(h^2 + \varepsilon^{1/2}h\right), \\[4pt]
(iii) & & \|u - \pi_2 u\|_{\infty,\Omega_c} \leq Ch^2, \\[4pt]
(iv) & & J(u - \pi_2 u, u - \pi_2 u)^{1/2} \leq Ch^{3/2} \\[4pt]
(v) & & \|\partial_n (u - \pi_2 u)\|_{0,\Gamma_c^-} \leq Ch^{1/2}.
\end{array}
$$

Proof. For $w \in H^2$ the error of the L_2 projection satisfies

$$
\|w - \pi_2 w\|_{0,\Omega_c} + h\,|w - \pi_2 w|_{0,\Omega_c} \leq Ch^2 \|w\|_{2,\Omega_c}. \tag{4.4}
$$

(i) Recalling the decomposition of $u = S + E$, $E = E_1 + E_2 + E_{12}$, we estimate

$$
\|u - \pi_2 u\|_{0,\Omega_c} \leq \|S - \pi_2 S\|_{0,\Omega_c} + \|E - \pi_2 E\|_{0,\Omega_c} \leq Ch^2 \|S\|_{2,\Omega_c} + \|E\|_{0,\Omega_c} \leq Ch^2,
$$

where we have used Lemma 4.1 and (2.1).

(ii) The result can be established by using

$$
\begin{aligned}
|u - \pi_2 u|_{1,\Omega_c} &\leq |S - \pi_2 S|_{1,\Omega_c} + |E - \pi_2 E|_{1,\Omega_c} \\
&\leq Ch \|S\|_{2,\Omega_c} + |E|_{1,\Omega_c} \leq C\left(h + \varepsilon^{-1/2}h^2\right),
\end{aligned}
$$

and by Lemma 4.1 and (2.1).

(iii) Use Lemma 4.1 to estimate as follows

$$
\|u - \pi_2 u\|_{\infty,\Omega_c} \leq \left\|u - u^I\right\|_{\infty,\Omega_c} + \left\|\pi_2 u^I - \pi_2 u\right\|_{\infty,\Omega_c} \leq C \left\|u - u^I\right\|_{\infty,\Omega_c}.
$$

The nodal interpolation error has been studied in, e.g. [11]. We then get the third bound.

(iv) Next consider $J(\cdot,\cdot)$. Clearly,

$$
J(u - \pi_2 u, u - \pi_2 u)^{1/2} \leq J(S - \pi_2 S, S - \pi_2 S)^{1/2} + J(E - \pi_2 E, E - \pi_2 E)^{1/2}
$$

and we bound the two terms separately. Let $\tau, \tau^* \in \mathcal{T}_c^N$ be the two mesh triangles with common edge $e \in \mathcal{E}_c^N$. Lemma 4.2 yields

$$
\int_e [\nabla(S - \pi_2 S)]^2 \leq C\left(h^{-1} \|\nabla(S - \pi_2 S)\|_{0,\tau\cup\tau^*}^2 + h \|S\|_{2,\tau\cup\tau^*}^2\right).
$$

Thus

$$
\begin{aligned}
\gamma_J h^2 \sum_{e \in \mathcal{E}_c^N} \int_e [\nabla(S - \pi_2 S)]^2 &\leq C\left(h \|\nabla(S - \pi_2 S)\|_{0,\Omega_c}^2 + h^3 \|S\|_{2,\Omega_c}^2\right) \\
&\leq Ch^3 \|S\|_{2,\Omega_c}^2 \leq Ch^3
\end{aligned}
$$

by (4.4) and (2.1).

Because $E \in C^1(\Omega_c)$

$$
\begin{aligned}
\int_e [\nabla(E - \pi_2 E)]^2 &= \int_e [\nabla(\pi_2 E)]^2 \\
&\leq Ch^{-1} \|\nabla(\pi_2 E)\|_{0,\tau\cup\tau^*}^2 \leq Ch^{-3} \|\pi_2 E\|_{0,\tau\cup\tau^*}^2,
\end{aligned}
$$

by Lemma 4.2 and an inverse inequality on τ. Summing over all edges, we get

$$\gamma_J h^2 \sum_{e \in \mathcal{E}_c^N} \int_e [\nabla(E - \pi_2 E)]^2 \leq Ch^{-1} \|\pi_2 E\|_{0,\Omega_c}^2 \leq Ch^{-1} \|E\|_{0,\Omega_c}^2 \leq Ch^3,$$

by Lemma 4.1, (2.1) and because $\sigma \geq 2$.

(v) Finally,

$$\|\partial_n(u - \pi_2 u)\|_{0,\Gamma_c^-}$$
$$\leq \|\partial_n(S - \pi_2 S)\|_{0,\Gamma_c^-} + \|\partial_n(\pi_2 E)\|_{0,\Gamma_c^-} + \|\partial_n E\|_{0,\Gamma_c^-}.$$

Lemma 4.2 yields for $e \subset \partial\tau$

$$\|\partial_n(S - \pi_2 S)\|_{0,e}^2 \leq C \left(h^{-1} \|S - \pi_2 S\|_{1,\tau}^2 + h \|S\|_{2,\tau}^2 \right).$$

Thus

$$\|\partial_n(S - \pi_2 S)\|_{0,\Gamma_c^-}^2$$
$$\leq C \left(h^{-1} \|S - \pi_2 S\|_{1,\Omega_c}^2 + h \|S\|_{2,\Omega_c}^2 \right) \leq Ch \|S\|_{2,\Omega_c}^2 \leq Ch,$$

by standard L_2-projection error bounds. Next,

$$\|\partial_n(\pi_2 E)\|_{0,e} \leq Ch^{-1/2} \|\pi_2 E\|_{1,\tau} \leq Ch^{-3/2} \|\pi_2 E\|_{0,\tau},$$

because $\pi_2 E$ is polynomial. Therefore

$$\|\partial_n(\pi_2 E)\|_{0,\Gamma_c^-} \leq Ch^{-3/2} \|\pi_2 E\|_{0,\Omega_c} \leq Ch^{1/2},$$

by Lemma 4.1 and (2.1). From the decomposition (2.1), we have

$$|\partial_n E(x,y)| \leq C \left[N^{-\sigma} + \varepsilon^{-1} \left(e^{-\beta_1/\varepsilon} + e^{-\beta_2/\varepsilon} \right) \right] \quad \text{for } (x,y) \in \Gamma_c^-.$$

For $h < \frac{1}{2}$ we have $\lambda_1 + h \leq 1$ and therefore

$$e^{-\beta_1/\varepsilon} \leq e^{-\beta_1(\lambda_1+h)/\varepsilon} \leq N^{-\sigma} e^{-\beta_1 h/\varepsilon} \leq C\varepsilon h^{\sigma-1},$$

which is easily verified since $\xi e^{-\beta_1 \xi}$ is bounded for $\xi \to \infty$. We obtain

$$\|\partial_n E\|_{0,\Gamma_c^-} \leq Ch^{\sigma-1}.$$

Collecting the results, we get the final estimate of the theorem. $\qquad\square$

4.2. Approximation error

Now we return to bounding the various error contributions in (4.1).

4.2.1. Coarse mesh region Ω_c

Here we have to bound, with $\eta = \pi_2 u - u$,

$$
\begin{aligned}
a\,(\eta, \chi)_{\Omega_c} &+ a_{bc}(\eta, \chi)_{\Gamma_c^{*-} \cap \partial\Omega_c} + J(\eta, \chi) \\
&= \varepsilon\,(\nabla\eta, \nabla\chi)_{\Omega_c} + (\eta, \boldsymbol{b} \cdot \nabla\chi)_{\Omega_c} + ((c + \mathrm{div}\boldsymbol{b})\eta, \chi)_{\Omega_c} + J(\eta, \chi) \\
&\quad + \int_{\Gamma_c^+} \boldsymbol{b} \cdot n\eta\chi - \varepsilon \int_{\Gamma_c^-} (\partial_n\eta\chi + \eta\partial_n\chi) + \gamma_{bc} \sum_{e \subset \Gamma_c^-} \frac{\varepsilon}{h_e} \int_e \eta\chi,
\end{aligned}
$$

where Γ_c^+ is the outflow boundary of Ω_c and Γ_c^- its inflow boundary.

Immediate consequences of the Cauchy-Schwarz inequality and of Theorem 4.1 are

$$
\varepsilon \left|(\nabla\eta, \nabla\chi)_{\Omega_c}\right| \le C\big(h^2 + \varepsilon^{1/2}h\big) \|\|\chi\|\|_\varepsilon\,,
$$

$$
\left|((c + \mathrm{div}\boldsymbol{b})\eta, \chi)_{\Omega_c}\right| \le Ch^2 \|\chi\|_0\,, \quad |J(\eta, \chi)| \le Ch^{3/2} J(\chi, \chi)^{1/2},
$$

$$
\gamma_{bc} \left| \sum_{e \subset \Gamma_c^-} \frac{\varepsilon}{h_e} \int_e \eta\chi \right| \le C \frac{\varepsilon^{1/2}}{h^{1/2}} \|\eta\|_{\infty, \Gamma_c^-} \left\| (\gamma_{bc}\varepsilon/h)^{1/2} \chi \right\|_{0, \Gamma_c^-}
$$

$$
\le C\left(h^{3/2} + \varepsilon^{1/2}h\right) \|\|\chi\|\|\,.
$$

An inverse inequality yields

$$
\varepsilon \left| \int_{\Gamma_c^-} \eta\partial_n\chi \right| \le \varepsilon \|\eta\|_{\infty, \Gamma_c^-} \|\partial_n\chi\|_{0, \Gamma_c^-}
$$

$$
\le C \frac{\varepsilon}{h} \|\eta\|_{\infty, \Gamma_c^-} \|\chi\|_{0, \Gamma_c^-} \le C\left(h^{3/2} + \varepsilon^{1/2}h\right) \|\|\chi\|\|\,.
$$

From Theorem 4.1(v) we get

$$
\varepsilon \left| \int_{\Gamma_c^-} \partial_n\eta\chi \right| \le C\varepsilon^{1/2}h^{1/2} \|\partial_n\eta\|_{0, \Gamma_c^-} \left\| (\gamma_{bc}\varepsilon/h)^{1/2} \chi \right\|_{0, \Gamma_c^-} \le C\varepsilon^{1/2}h \|\|\chi\|\|\,.
$$

Next Theorem 4.1(iii) gives

$$
\left| \int_{\Gamma_c^+} \boldsymbol{b} \cdot n\eta\chi \right| \le Ch^2 \|\chi\|_{0, \Gamma_c^+} \le Ch^{3/2} \|\chi\|_0,
$$

since $\|w\|_{0,e} \le Ch^{-1/2} \|w\|_{0,\tau}$ for polynomials w and any $e \subset \partial\tau$ with $\tau \in \mathcal{T}_c^N$.

Now only the term $(\eta, \boldsymbol{b} \cdot \nabla\chi)_{\Omega_c}$ remains to be estimated. For this purpose the following auxiliary result will be used.

Lemma 4.3. *Let $\tilde{V}_c^h$ be the space of piecewise linear, but not necessarily continuous functions over $\mathcal{T}_c^N$. Then there exists a projection operator $\pi^* : H^2(\Omega_c) \to V_c^h$ and a constant C, such that*

$$
\|p - \pi^*p\|_{0,\Omega_c}^2 \le Ch \sum_{e \in \mathcal{E}_c^N} [p]^2 \quad \forall p \in \tilde{V}_c^h.
$$

Proof. See [4].   □

We proceed as follows. With $\boldsymbol{b}^I$ denoting the piecewise linear nodal interpolant of $\boldsymbol{b}$, we have

$$(\eta, \boldsymbol{b} \cdot \nabla\chi)_{\Omega_c} = \left(\eta, (\boldsymbol{b} - \boldsymbol{b}^I) \cdot \nabla\chi\right)_{\Omega_c} + \left(\eta, \boldsymbol{b}^I \cdot \nabla\chi\right)_{\Omega_c}. \tag{4.5}$$

The first term is easily bounded by

$$\left|\left(\eta, (\boldsymbol{b} - \boldsymbol{b}^I) \cdot \nabla\chi\right)_{\Omega_c}\right| \le Ch^4 \|\nabla\chi\|_{\Omega_c} \le Ch^3 \|\chi\|_{\Omega_c},$$

by Theorem 4.1(i), standard interpolation error bounds for $\boldsymbol{b} - \boldsymbol{b}^I$ and an inverse inequality.

For the second term in (4.5) use the orthogonality of the L_2 projection

$$\left|\left(\eta, \boldsymbol{b}^I \cdot \nabla\chi\right)_{\Omega_c}\right| = \left|\left(\eta, \boldsymbol{b}^I \cdot \nabla\chi - \pi^*(\boldsymbol{b}^I \cdot \nabla\chi)\right)_{\Omega_c}\right|$$

$$\le Ch^{5/2} \left(\sum_{e \in \mathcal{E}_c^N} [\boldsymbol{b}^I \cdot \nabla\chi]^2\right)^{1/2} \le Ch^{3/2} J(\chi, \chi)^{1/2}$$

by Lemma 4.3 and Theorem 4.1(i).

Summarizing the results of this section, we have

$$\left|a(\eta, \chi)_{\Omega_c} + a_{bc}(\eta, \chi)_{\Gamma_c^{*-} \cap \partial\Omega_c} + J(\eta, \chi)\right| \le C\left(\varepsilon^{1/2}h + h^{3/2}\right) |||\chi|||_E. \tag{4.6}$$

4.2.2. Transition region Ω_t.

Let us consider $a(\pi u - u^I, \chi)_{\Omega_t}$ and $a_{bc}(\pi u - u, \chi)_{\Gamma_c^{*-} \cap \partial\Omega_t}$.

First, note that

$$\left\|\pi u - u^I\right\|_{\infty, \Omega_t} \le CN^{-2} \tag{4.7}$$

because the piecewise bilinear function $\pi u - u^I$ vanishes at the outflow boundary of Ω_t and therefore it can be bounded by the maximum of $\pi_2 u - u^I$ on Γ_c^+ for which an upper bound is provided by Theorem 4.1(iii).

Let $\tilde{\Gamma} := \Gamma \cap \Gamma_t$ which consists of just two edges in the layer region next to the transition line. By $\tilde{\Omega}$ we denote domain formed by the two elements $\tau \in \mathcal{T}_f^N$ adjacent to $\tilde{\Gamma}$. Then

$$a(\pi u - u^I, \chi)_{\Omega_t} + a_{bc}(\pi u - u, \chi)_{\Gamma_c^{*-} \cap \partial\Omega_t}$$

$$= \varepsilon\left(\nabla(\pi u - u^I), \nabla\chi\right)_{\Omega_t} + \left(\pi u - u^I, (c + \operatorname{div}\boldsymbol{b})\chi + \boldsymbol{b} \cdot \nabla\chi\right)_{\Omega_t}$$

$$- \left(\boldsymbol{b} \cdot n(\pi u - u^I), \chi\right)_{\Gamma_c^+} + \left(\boldsymbol{b} \cdot n(u^I - u), \chi\right)_{\tilde{\Gamma}}$$

$$- \varepsilon\left(\partial_n(\pi u - u), \chi\right)_{\tilde{\Gamma}} + \varepsilon\left(\pi u - u, \partial_n\chi\right)_{\tilde{\Gamma}} + \gamma_{bc} \sum_{e \subset \tilde{\Gamma}} \frac{\varepsilon}{h_e}(\pi u - u, \chi)_e$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.$$

(I_1) An inverse inequality and (4.7) give

$$\left\|\nabla(\pi u - u^I)\right\|_{0, \Omega_t} \le C\varepsilon^{-1/2} N^{-3/2} \ln^{-1/2} N.$$

Thus

$$|I_1| \le CN^{-3/2} |||\chi|||_\varepsilon.$$

(I_2) By (4.7)

$$\left\|\pi u - u^I\right\|_{0,\Omega_t} \leq C\varepsilon^{1/2} N^{-5/2} \ln^{1/2} N$$

and therefore

$$|I_2| \leq C N^{-5/2} \ln^{1/2} N \left\|\left|\chi\right|\right\|_\varepsilon.$$

(I_3) Again we start from (4.7):

$$\left\|\pi u - u^I\right\|_{0,\Gamma_c^+} \leq C N^{-2}.$$

A discrete trace inequality yields

$$\left\|\chi\right\|_{0,\Gamma_c^+} \leq C h^{-1/2} \left\|\chi\right\|_{0,\Omega_c}.$$

Thus

$$|I_3| \leq C N^{-3/2} \left\|\chi\right\|_{0,\Omega_c}.$$

(I_4) The interpolation error bound (4.3) gives

$$\left\|u^I - u\right\|_{0,\tilde\Gamma} \leq C\varepsilon^{1/2} N^{-5/2} \ln^{5/2} N,$$

while by a discrete trace inequality we obtain

$$\left\|\chi\right\|_{0,\tilde\Gamma} \leq C h^{-1/2} \left\|\chi\right\|_{0,\tilde\Omega}. \tag{4.8}$$

Hence

$$|I_4| \leq C N^{-2} \ln^2 N \left\|\chi\right\|_{0,\Omega} \leq C N^{-3/2} \left\|\chi\right\|_{0,\Omega}.$$

(I_5) First let us derive a pointwise bound for $\partial_n(\pi u - u)$. By a triangle inequality

$$\left\|\partial_n(\pi u - u)\right\|_{\infty,\tilde\Gamma} \leq \left\|\partial_n(\pi u - u^I)\right\|_{\infty,\tilde\Gamma} + \left\|\partial_n(u^I - u)\right\|_{\infty,\tilde\Gamma}.$$

The step size normal to $\tilde\Gamma$ is $\mathcal{O}\left(N^{-1}\right)$. Therefore and by (4.7)

$$\left\|\partial_n(\pi u - u^I)\right\|_{\infty,\tilde\Gamma} \leq C N^{-1}.$$

Next let us consider $\partial_n(u^I - u)$ on the edge $\tilde e = [x_{N/2-1}, x_{N/2}] \times \{1\}$, with the adjacent mesh rectangle $\tilde\tau$. Here $\partial_n = \partial_y$. Therefore

$$\left\|\partial_n(u^I - u)\right\|_{\infty,\tilde e} \leq \left\|\partial_y(u^I - u)\right\|_{\infty,\tilde\tau} \leq C\left(N^{-1} \left\|\partial_{yy} u\right\|_{\infty,\tilde\tau} + \varepsilon N^{-1} \ln N \left\|\partial_{xy} u\right\|_{\infty,\tilde\tau}\right),$$

by an anisotropic interpolation error bound [1]. By (2.1) we have

$$\left\|\partial_{yy} u\right\|_{\infty,\tilde\tau} \leq C \quad \text{and} \quad \left\|\partial_{xy} u\right\|_{\infty,\tilde\tau} \leq C\left(1 + \varepsilon^{-1} N^{-\sigma}\right).$$

Hence

$$\left\|\partial_n(u^I - u)\right\|_{\infty,\tilde e} \leq C N^{-1}$$

with an identical bound for the other edge in $\tilde{\Gamma}$. Thus

$$\|\partial_n(\pi u - u)\|_{\infty,\tilde{e}} \leq CN^{-1}.$$

Integrating over $\tilde{\Gamma}$, we get

$$\left\|\partial_n(\pi u - u^I)\right\|_{0,\tilde{\Gamma}} \leq C\varepsilon^{1/2}N^{-3/2}\ln^{1/2}N.$$

Use (4.8) in order to obtain

$$\varepsilon\left|\left(\partial_n(\pi u - u^I), \chi\right)_{\tilde{\Gamma}}\right| \leq C\varepsilon^{1/2}N^{-1}\|\chi\|_0.$$

(I_6) A triangle inequality, (4.7) and (4.3) imply

$$\|\pi u - u\|_{\infty,\tilde{\Gamma}} \leq \left\|\pi u - u^I\right\|_{\infty,\tilde{\Gamma}} + \left\|u^I - u\right\|_{\infty,\tilde{\Gamma}} \leq CN^{-2}\ln^2 N.$$

Thus

$$\|\pi u - u\|_{0,\tilde{\Gamma}} \leq C\varepsilon^{1/2}N^{-5/2}\ln^{5/2}N.$$

Furthermore, by a discrete trace inequality

$$\|\partial_n\chi\|_{0,\tilde{\Gamma}} \leq Ch^{-1/2}\|\nabla\chi\|_{0,\tilde{\Omega}}.$$

Thus

$$|I_6| \leq C\varepsilon N^{-2}\ln^{5/2}N\,\|\|\chi\|\|_\varepsilon.$$

(I_7) By (4.8)

$$|I_7| \leq C\varepsilon N^{-2}\ln^{5/2}N\left\|(\gamma_{bc}\varepsilon/h)^{1/2}\chi\right\|_{0,\tilde{\Gamma}} \leq C\varepsilon N^{-2}\ln^{5/2}N\,\|\|\chi\|\|.$$

Collecting the various estimates of this section, we arrive at

$$\left|a\left(\pi u - u^I, \chi\right)_{\Omega_t} + a_{bc}(\pi u - u, \chi)_{\Gamma_c^{*-}\cap\partial\Omega_t}\right| \leq \left(\varepsilon^{1/2}N^{-1} + N^{-3/2}\right)\|\|\chi\|\|. \qquad (4.9)$$

4.2.3. Global error

Combining the estimates (4.1), (4.2), (4.6) and (4.9) we obtain our final error bound upon dividing by $\|\|\chi\|\|_E$.

Theorem 4.2. *Assume the solution u of (1.1) satisfies (2.1). Let u^h be the solution of (3.2) on a Shishkin mesh with $\sigma \geq 2$. Then*

$$\|\|\pi u - u^h\|\|_E \leq C\left(\varepsilon^{1/2}N^{-1} + N^{-3/2}\right).$$

Remark 4.1. Combing Theorem 4.2 with projection error estimates of Section 4.1, we get

$$\|\|\pi u - u^h\|\|_E \leq CN^{-1}\ln N. \qquad (4.10)$$

Thus, Theorem 4.2 is a supercloseness result for the mixed nodal/L_2 projection of the exact solution. It can be used to construct postprocessing procedures to obtain improved estimates for the derivatives than those provided by (4.10).

Remark 4.2. Numerical experiments for edge stabilization with purely bilinear elements on tensor-product Shishkin meshes were presented in [9], a byproduct of the present paper. [9] also contains a simplified convergence analysis for that special case.

A number of interesting phenomena of the finite element method on hybrid meshes were observed during our experiments. These numerical results extend beyond the scope of the present study and will therefore be discussed in detail in a forthcoming paper.

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