Journal of Computational Mathematics
Vol.28, No.2, 2010, 273–288.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.10-m2870

# UNIFORM SUPERCONVERGENCE OF GALERKIN METHODS FOR SINGULARLY PERTURBED PROBLEMS*

Ying Chen
*Department of Mathematics, University of California, Irvine, CA 92697, USA*
*Email: yingc@uci.edu*
Min Huang
*Department of Computer Science, Purdue University, West Lafayette, IN 47907, USA*
*Email: huang3@purdue.edu*

**Abstract**

In this paper, we are concerned with uniform superconvergence of Galerkin methods for singularly perturbed reaction-diffusion problems by using two Shishkin-type meshes. Based on an estimate of the error between spline interpolation of the exact solution and its numerical approximation, an interpolation post-processing technique is applied to the original numerical solution. This results in approximation exhibit superconvergence which is uniform in the weighted energy norm. Numerical examples are presented to demonstrate the effectiveness of the interpolation post-processing technique and to verify the theoretical results obtained in this paper.

*Mathematics subject classification:* 65L10, 65L60.
*Key words:* singularly perturbed, Hermite splines, Shishkin-type meshes, Interpolation post-processing, Uniform superconvergence.

## 1. Introduction

In this paper, we consider the singularly perturbed two-point boundary value problem of reaction-diffusion type. It is well-known that the solution of this problem exhibits singularities at the boundary layers where singularities depend upon perturbation parameters. When the problem is solved numerically, we must take this boundary layer behavior of the solution into account in order to produce an approximate solution with high-order convergence. Shishkin meshes are most commonly used meshes in numerical methods which include finite difference methods and finite element methods (see, e.g., [9, 10, 12, 13] and references cited therein). In [12,13], the finite element method on the standard Shishkin mesh ($S$-mesh) provided a numerical solution with convergence rate which is almost optimal uniformly in the weighted energy norm. Based on another Shishkin-type mesh, namely, the anisotropic mesh ($A$-mesh), Li [3] proved an optimal order of uniform convergence for high-order reaction-diffusion problems. One of the advantages of Shishkin-type meshes, broadly defined, is that they are piecewise equidistant meshes. This structure of Shishkin-type meshes can be exploited and we show in this paper that, when it is combined with the interpolation post-processing technique, uniform superconvergence of numerical solution can be obtained.

Therefore, the main purpose of this paper is to obtain uniform superconvergence in the weighted energy norm of the Galerkin method on $S$-mesh as well as on $A$-mesh for singularly

* Received July 9, 2008 / Revised version received May 11, 2009 / Accepted June 8, 2009 / Published online December 21, 2009 /

perturbed reaction-diffusion problems. Previously, Zhang [14] investigated a superconvergence phenomena of a finite element solution obtained on a modified Shiskin mesh for the second-order singularly perturbed problems. The superconvergence estimate was given in a *discrete* weighted energy norm in which the $L^2$ norm of $\nabla(u-u_h)$ is replaced by a $p$-point Gaussian quadrature rule. In this paper, motivated by an estimate of the error between spline interpolation of the exact solution and its numerical approximation, which is commonly known as *superclose*, (see, e.g., [2,4,5]), we will apply the interpolation post-processing technique to the numerical solution obtained by the Galerkin method. This generates a higher order approximation which gives rise to uniform superconvergence in the weighted energy norm. It will be shown that we may gain improvements in the order of convergence in the following way. For $S$-mesh, the rate of convergence in the weighted energy norm is enhanced from the optimal order of convergence *up to a logarithmic factor* and for $A$-mesh, the order of convergence improves by one from the optimal. We point out that the superclose property mentioned above plays a key role in establishing the superconvergence result presented in this paper. The idea of superclose property was used to obtain numerical solution of different operator equations. For one dimensional problem, we refer the reader to [3,8,12] and for two dimensional problems [6,7]. Also, we point out that fact that the idea of the interpolation post-processing technique has been successfully used by several authors (see, [2,4,5]). Finally, we note that our current work can be extended to two-dimensional reaction-diffusion problems as well as to other singularly perturbed problems. These topics will be discussed in the forthcoming papers.

This paper is organized as follows. In section 2, we describe the Galerkin method on $S$-mesh as well as on $A$-mesh for solving high-order reaction-diffusion problems. Section 3 is devoted to a study of application of the interpolation post-processing technique. The post-processing is applied to the original numerical solution of high-order reaction-diffusion problems. The uniform superconvergence of the post-processed solution on $S$-mesh as well as on $A$-mesh is subsequently obtained in the weighted energy norm. In section 4, we consider the second-order reaction-diffusion problem. The superclose property of the approximate solution in the weighted energy norm is derived by constructing a special interpolant. Based on this estimate, we use the interpolation post-processing technique to achieve the uniform superconvergence property of the numerical solution. Finally in section 5, two numerical examples are presented to confirm the theoretical results obtained in the previous sections.

## 2. High-order Reaction-Diffusion Problems

We introduce in this section the Galerkin method on $S$-mesh as well as on $A$-mesh for solving reaction-diffusion problems using Hermite splines. We begin with some notations. Set $\mathbb{N} := \{1,2,...\}$, $\mathbb{N}_0 := \{0,1,...\}$ and $\mathbb{Z}_n := \{0,1,...,n-1\}$. Let $I := [0,1]$ and $T$ be a subinterval of $I$. We denote by $(\cdot,\cdot)_T$ the inner product in $L^2(T)$ and by $\|\cdot\|_{0,T}$ for the associated norm on $L^2(I)$. Let $H^k(T), k \in \mathbb{N}$, be the Sobolev spaces on $T$ with the norm $\|\cdot\|_{k,T}$ defined by

$$\|v\|_{k,T} = \left\{ \sum_{i=0}^{k} \int_T |v^{(i)}(x)|^2 dx \right\}^{1/2},$$

and the semi-norm $|\cdot|_{k,T}$ defined by

$$|v|_{k,T} = \left\{ \int_T |v^{(k)}(x)|^2 dx \right\}^{1/2}.$$