Journal of Computational Mathematics
Vol.28, No.3, 2010, 289–308.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1001-m1002

# FRAMELET BASED DECONVOLUTION*

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**Abstract**

In this paper, two framelet based deconvolution algorithms are proposed. The basic idea of framelet based approach is to convert the deconvolution problem to the problem of inpainting in a frame domain by constructing a framelet system with one of the masks being the given (discrete) convolution kernel via the unitary extension principle of [26], as introduced in [6–9] . The first algorithm unifies our previous works in high resolution image reconstruction and infra-red chopped and nodded image restoration, and the second one is a combination of our previous frame-based deconvolution algorithm and the iterative thresholding algorithm given by [14, 16]. The strong convergence of the algorithms in infinite dimensional settings is given by employing proximal forward-backward splitting (PFBS) method. Consequently, it unifies iterative algorithms of infinite and finite dimensional setting and simplifies the proof of the convergence of the algorithms of [6].

*Mathematics subject classification:* 65T60, 90C90, 94A08
*Key words:* Framelet, deconvolution, wavelet, tight frame, soft-thresholding

## 1. Introduction

The deconvolution is to solve $\mathbf{v}$ given by the following convolution equation:

$$\mathbf{c} = \mathbf{h} * \mathbf{v} + \epsilon, \tag{1.1}$$

where $\mathbf{h}$, $\mathbf{c}$ and $\epsilon$ are all in $\ell_2(\mathbb{Z})$, and $*$ is the convolution operator. The sequence $\mathbf{h}$ is the blurring kernel, and $\mathbf{c}$ is the observed signal. The sequence $\epsilon$ is the error term satisfying $\|\epsilon\|_{\ell_2(\mathbb{Z})} \leq \varepsilon$. The basic idea of our framelet based approach is to convert the deconvolution problem to the problem of inpainting in a frame domain by constructing a framelet system with one of the masks being the given (discrete) convolution kernel $\mathbf{h}$ via the unitary extension principle of [26].

This framelet based approach for deconvolution was originally proposed in [7–9] for high-resolution image reconstruction, by using frames derived from bi-orthogonal wavelets or the unitary extension principle of [26]. It was then extended to video still enhancement [11] and to infrared image restoration [3]. Recently, this framelet based approach is further generalized to inpainting in the image domain by [4,10]. The numerical simulation results in those papers show clearly that this framelet based approach is numerically efficient and easy to implement. The framelet deconvolution algorithm was first analyzed in [6]. This paper is to unify the framelet based approaches in the literature and to give a complete analysis of the unified approach.

* Received December 31, 2008 / Revised version received May 20, 2009 / Accepted June 6, 2009 / Published online February 1, 2010 /

There are several papers on solving inverse problems, in particular deconvolution problems, by using wavelet methods. The wavelet-vaguelette decomposition methods by [18, 21], the deconvolution in mirror wavelet bases by [24, 25], Galerkin-type methods to inverse problems using an appropriate basis by [1, 12], and the orthonormal wavelet method by [14, 18] are examples of wavelet approaches. The connections and differences of the above approaches and the framelet approaches of [3, 7–9, 11] are detailed in [6], and interested readers should consult [6] for the details. Finally, we also refer the reader to the recent work of [5] that gives a different approach on framelet based deconvolution by using linearized Bregman iteration.

In this paper, we propose and analyze two algorithms. The first one unifies the previous works in high resolution image reconstruction [7–9] and infra-red chopped and nodded image restoration [3], and the second one is a combination of the frame-based deconvolution algorithm [6] and the iterative thresholding algorithm of [14, 16]. The strong convergence of the algorithms is given in infinite dimensional setting by employing proximal forward-backward splitting (PFBS) method [13]. Consequently, it unifies iterative algorithms of infinite and finite dimensional setting, simplifies the proof of the convergence of the algorithms, and improves the minimization results that the limits are satisfied, of [6].

Since the focus of this paper is to give a theoretical analysis of algorithms, the numerical simulation is not the focus of this paper. The interested readers should refer to [3, 7–9, 11] for the numerical simulations for various applications.

The paper is organized as follows. In Section 2, we give a review of framelets. In Section 3, we give algorithms for the framelet deconvolution approach. In Sections 4.1 and 4.2, we analyze the strong convergence of the algorithms by the theory of proximal forward-backward splitting. The corresponding results in finite dimensional setting are illustrated in Section 5.

## 2. Framelets

In this section, we review some of basics of framelet that are needed for the current paper. For those who are familiar with the notion of framelet may skip this section.

A countable set $X \subset L_2(\mathbb{R})$ is called a *tight frame* of $L_2(\mathbb{R})$ if

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{g \in X} |\langle f, g\rangle|^2,$$

or equivalently

$$f = \sum_{g \in X} \langle f, g\rangle\, g,$$

holds for all $f \in L_2(\mathbb{R})$, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_{L_2(\mathbb{R})}$ are the inner product and the norm in $L_2(\mathbb{R})$ respectively. For given $\Psi := \{\psi_1, \dots, \psi_r\} \subset L_2(\mathbb{R})$, the *affine system* is defined by

$$X(\Psi) := \Big\{\psi_{\ell,j,k} : 1 \le \ell \le r;\ j, k \in \mathbb{Z}\Big\} \quad \text{with} \quad \psi_{\ell,j,k} := 2^{j/2}\psi_\ell(2^j \cdot -k).$$

When $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R})$, it is called a *tight wavelet frame*, and $\psi_\ell$, $\ell = 1, \dots, r$, are called the *tight framelets*.

The *quasi-affine system* from level $J$ is defined as

$$X^q_J(\Psi) = \Big\{\psi^q_{\ell,j,k} :\ 1 \le \ell \le r; j, k \in \mathbb{Z}\Big\} \quad \text{with} \quad \psi^q_{\ell,j,k} := \begin{cases} 2^{j/2}\psi_\ell(2^j \cdot -k), & j \ge J; \\ 2^{j-\frac{J}{2}}\psi_\ell(2^j \cdot -2^{j-J}k), & j < J. \end{cases}$$