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ON THE FINITE ELEMENT APPROXIMATION OF SYSTEMS OF REACTION-DIFFUSION EQUATIONS BY p/hp METHODS*

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Abstract

We consider the approximation of *systems* of reaction-diffusion equations, with the finite element method. The highest derivative in each equation is multiplied by a parameter $\varepsilon \in (0,1]$, and as $\varepsilon \to 0$ the solution of the system will contain *boundary layers*. We extend the analysis of the corresponding scalar problem from [Melenk, IMA J. Numer. Anal. 17(1997), pp. 577-601], to construct a finite element scheme which includes elements of size $\mathcal{O}(\varepsilon p)$ near the boundary, where p is the degree of the approximating polynomials. We show that, under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε, when the error is measured in the energy norm associated with the problem. Numerical computations supporting the theory are also presented, which also show that the method yields robust exponential convergence rates when the error in the maximum norm is used.

Mathematics subject classification: 65N30.
Key words: Reaction-diffusion system, Boundary layers, *hp* finite element method.

1. Introduction

The numerical solution of reaction-diffusion problems whose solution contains boundary layers has been studied extensively over the last two decades (see, e.g., the books [5, 6, 8] and the references therein). The presence of boundary layers in the solution cannot be overlooked, and if one wishes to obtain an accurate and robust approximation, special care must be taken when constructing the numerical method. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [11] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [10]. In both cases, the a-priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [2, 4, 10].

In recent years researchers have turned their attention to *systems* of reaction-diffusion problems — see [3] and the references therein for a recent survey. In general, one-dimensional reaction diffusion systems, like the one considered in the present article, have the following

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form: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} \equiv \begin{bmatrix} -\varepsilon_1^2 \frac{d^2}{dx^2} & & 0 \\ & \ddots & \\ 0 & & -\varepsilon_m^2 \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{u} + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{1.1}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}, \tag{1.2}$$

where $0 < \varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_m \leq 1$,

$$A = \begin{bmatrix} a_{11}(x) & ... & a_{1m}(x) \\ \vdots & & \vdots \\ a_{m1}(x) & ... & a_{mm}(x) \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}. \tag{1.3}$$

The data $\{\varepsilon_i\}_{i=1}^m$, A and $\overrightarrow{f}$ are given, and the unknown solution is $\overrightarrow{u}(x) = [u_1(x), ..., u_m(x)]^T$. The functions $a_{ij}(x)$ are such that for any $x \in \overline{\Omega} = [0,1]$, the matrix A is invertible (with $\|A^{-1}\|$ bounded) and moreover

$$\overrightarrow{\xi}^T A \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^m, \tag{1.4}$$

for some constant $\alpha > 0$.

We will restrict ourselves to the case $\varepsilon_i = \varepsilon \ \forall\ i = 1, ..., m$, which allows us to express (1.1)–(1.2) in vector form as: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} := -\varepsilon^2 \overrightarrow{u}'' + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{1.5}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}. \tag{1.6}$$

The presence of the small parameter ε in the above boundary value problem causes the solution $\overrightarrow{u}$ to contain boundary layers of width $\mathcal{O}(|\varepsilon \ln \varepsilon|)$ near the endpoints of Ω. To illustrate this, we consider the case $m = 2$ with

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \varepsilon = 10^{-2}.$$

Figure 1.1 shows the exact solution corresponding to the above data and clearly shows that both components contain a boundary layer.

Our goal in the present article is to extend the analysis of [4] for the analogous scalar problem, to show that under the assumption of analytic input data, the *hp* version of the FEM on the variable three element mesh $\Delta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon\}, \kappa \in \mathbb{R}^+$ converges at an *exponential* rate (in the energy norm defined in eq. (2.6) below) as the polynomial degree of the approximating basis functions $p \to \infty$. Strictly speaking, the method is not an *hp* version, since the location and not the number of elements changes as the dimension of the approximating subspace is increased; a more appropriate characterization would be a p version on a variable mesh. In addition to extending the results of [4] to systems, our proof does not use Gauss-Lobatto interpolants (like the one in [4]), but rather we achieve the desired result using the approximation theory from [9] with integrated Legendre polynomials, something that is of interest in its own right. More importantly, the present approach allows us to define the constant κ used in the mesh in a more concrete way.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the properties of its solution. In Section 3 we present the finite element formulation and the design of the p/hp scheme we will be considering, along with our main result of exponential