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ESTIMATOR COMPETITION FOR POISSON PROBLEMS*

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Abstract

We compare 13 different a posteriori error estimators for the Poisson problem with lowest-order finite element discretization. Residual-based error estimators compete with a wide range of averaging estimators and estimators based on local problems. Among our five benchmark problems we also look on two examples with discontinuous isotropic diffusion and their impact on the performance of the estimators. (Supported by DFG Research Center MATHEON.)

Mathematics subject classification: 65N30, 65R20, 73C50.
Key words: Finite element methods, A posteriori error estimators.

1. Introduction

A posteriori error control has become an important issue for reliable and efficient computation of PDEs [1–6]. This paper updates the empirical study of [7] to modern a posteriori error control via the five classes of 13 estimators of Table 1.1 applied to the five benchmark examples of Table 1.2 such as the Poisson model problem on the L-shaped domain illustrated in Figure 1.1. Up to modified boundary conditions, marked by BC, all the benchmark problems are of the following type with or without discontinuous coefficients $\varkappa$ for some given right-hand side $f \in L^2(\Omega)$ and finite element approximation u_h to the unknown exact solution $u \in H^1_0(\Omega)$ of

$$\operatorname{div}(\varkappa \nabla u) + f = 0 \quad \text{in } \Omega. \tag{1.1}$$

Here and throughout the paper, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$, and the piecewise constant diffusion coefficient $\varkappa$ is bounded by

$$0 < \varkappa_{\min} \leq \varkappa(x) \leq \varkappa_{\max} < \infty \qquad \text{for all } x \in \overline{\Omega}. \tag{1.2}$$

By definition, an *error estimator* η is a computable quantity that aims to estimate the error $e := u - u_h$, e.g., in its energy norm,

$$|||e||| := \|\varkappa^{1/2} \nabla (u - u_h)\|_{L^2(\Omega)}.$$

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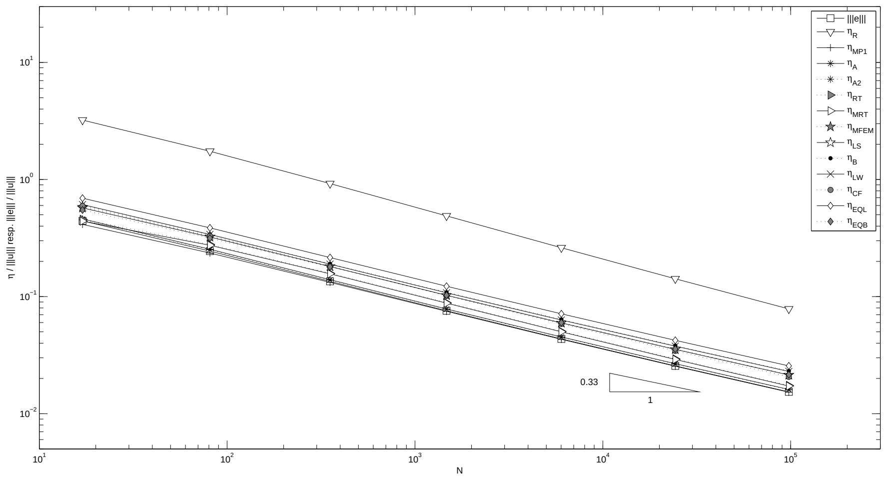

Fig. 1.1. Error and error estimators for uniform mesh refinement of L-shaped domain with right-hand side 1 from Example 7.1 in Section 7 to illustrate different accuracy of different error estimators.

Desirable properties of η are its *reliability* in the sense of an upper bound

$$|||e||| \leq \mathrm{C_{rel}}\eta + \text{h.o.t.}$$

and its *efficiency* in the sense of a lower bound

$$\eta \leq \mathrm{C_{eff}}|||e||| + \text{h.o.t.}$$

Any complete error control requires estimates of the constants $\mathrm{C_{rel}}$ and $\mathrm{C_{eff}}$ and the higher-order terms h.o.t. which are oscillations of the right-hand side f that are of magnitudes smaller than the energy error in all the examples of this paper. In many cases only the constant $\mathrm{C_{rel}} = 1$ is known while $\mathrm{C_{eff}}$ depends on generic constants [1, 3, 5].

We assume that $\mathcal{T}$ is a regular triangulation of Ω in the sense of Ciarlet [8, 9] with nodes $\mathcal{N}$, free nodes $\mathcal{K} = \mathcal{N}\backslash\partial\Omega$ and edges $\mathcal{E}$ such that $\varkappa \in \mathcal{P}_0(\mathcal{T})$. The discrete space $\mathcal{P}_k(\mathcal{T})$ denotes the $\mathcal{T}$-piecewise polynomials of degree $\leq k$. The nodal basis function associated to $z \in \mathcal{N}$ is denoted

Table 1.1: Classes of a posteriori error estimators studied in this paper.

No	Class error estimators	Examples (Reference below)
1	explicit residual-based	η_{R} (Section 2)
2	averaging	η_{A1}, η_{A2}, η_{MP1}, η_{RT}, η_{MRT} (Section 3)
3	equilibration	η_{B}, η_{MFEM}, η_{LW}, η_{EQL}, η_{EQB} (Section 4)
4	least-square	η_{LS} (Section 4.2)
5	localisation	η_{CF} (Section 5)

Table 1.2: Benchmark examples studied in this paper.

No	Short name	Problem description in (1.1)	Feature
1	L-shaped domain	$\varkappa \equiv f \equiv 1$	corner singularity
2	Square domain	$\varkappa \equiv 1$, f with oscillations	oscillations
3	Slit domain	$\varkappa \equiv f \equiv 1$ & BC	slit singularity
4	Interface problem	jumping $\varkappa$, $f \equiv 0$ & BC	interface singularity
5	Octagon example	jumping $\varkappa$, $f \equiv 0$ & BC	continuous fluxes