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EQUIVALENCE OF SEMI-LAGRANGIAN AND LAGRANGE–GALERKIN SCHEMES UNDER CONSTANT ADVECTION SPEED*

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Abstract

We compare in this paper two major implementations of large time-step schemes for advection equations, i.e., Semi-Lagrangian and Lagrange–Galerkin techniques. We show that SL schemes are equivalent to exact LG schemes via a suitable definition of the basis functions. In this paper, this equivalence will be proved assuming some simplifying hypoteses, mainly constant advection speed, uniform space grid, symmetry and translation invariance of the cardinal basis functions for interpolation. As a byproduct of this equivalence, we obtain a simpler proof of stability for SL schemes in the constant-coefficient case.

Mathematics subject classification: 65N12, 65M10, 49L25.
Key words: Semi-Lagrangian schemes, Lagrange–Galerkin schemes, Stability.

1. Introduction

High-order, large time-step schemes for first order equation have gained an increasing popularity in the last two decades. Schemes based on characteristics for hyperbolic PDEs have been proposed by Courant–Isaacson–Rees in [3] and have prompted the development of a number of specific techniques, including Semi-Lagrangian (SL) methods, which have first appeared in the framework of Numerical Weather Prediction problems (see the pioneering paper [18] and the review [15]). SL methods have another well estabilished field of application in plasma physics, where their use has been suggested in [2] and widely studied since (see, e.g., [1,8,16]), and their popularity is also growing in other fields such as general Computational Fluid Dynamics (CFD) problems, Hamilton–Jacobi equations and level-set methods [7,17], conservation laws [13]. On the other hand, Lagrange–Galerkin (LG) methods have been proposed independently in [5,12], and currently their main field of application is CFD (including Numerical Weather Prediction) in a finite element setting.

The main ideas of this work will be sketched focusing on the model problem of the constant-coefficient, evolutive advection equation

$$\begin{cases} v_t(x,t) + a \cdot \nabla v(x,t) = 0, & \text{in } \mathbb{R}^N \times \mathbb{R} \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

in which the transport velocity is given by the constant vector $a = (a_1 \cdots a_N)^T \in \mathbb{R}^N$.

The construction of large time-step schemes for (1.1) stems from the application of the method of characteristics (see, e.g., [5,6,12]), which uses the property of the solution of (1.1)

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to be constant along the characteristics lines $(x - a\tau, t - \tau)$ of the $x - t$ space. This means that the following time–discrete representation formula

$$v(x,t) = v(x - a\Delta t, t - \Delta t) \tag{1.2}$$

holds for the solution v.

In large time-step schemes the discretization is performed on the representation formula (1.2) rather than on the Eq. (1.1). The discretization of (1.2) is obtained by introducing a numerical reconstruction to approximate the value $v(x - a\Delta t, t - \Delta t)$, since in general the foot of the characteristic $x_j - a\Delta t$ does not coincide with any grid point (we note that in the more general setting of variable coefficient equations, characteristics are no longer straight lines, and the position of the foot of characteristics needs itself to be approximated).

As we will recall in the next section, the space reconstruction is precisely what discriminates between SL and LG schemes, and is also the crucial point in proving stability of the scheme, which is not a trivial result when using high-order techniques. In this respect, LG schemes allow for a cleaner and more general analysis, whereas high-order SL schemes have only be proved to be stable in the Von Neumann sense. Therefore, the equivalence of the two formulations (whenever provable) is also a way of obtaining a more general stability result for SL schemes.

In this paper, we will prove such an equivalence for the simplified case of (1.1), although in fact L^2 stability of SL schemes is a known fact for such a model – so, while expecting that this technique may also work in greater generality, this paper only presents a simpler stability proof, along with a deeper insight in the relationship between the two classes of schemes.

The paper is organized as follows. Section 2 reviews the construction and basic convergence theory of SL and LG schemes for (1.1), and section 3 sets up a framework in which the equivalence of SL and LG schemes can be proved. In section 4, we give some practical examples of reconstruction bases falling in this theory.

2. A Comparison of Semi-Lagrangian and Lagrange–Galerkin Schemes

This section presents the main ideas in the construction of SL and LG schemes. We point out that this is presentation is focused on the *basic* form of the schemes. Recent years have witnessed a number of developments, especially aimed at improving the performances of the schemes in presence of discontinuous solutions. Among such advances, we mention the use of nonlinear (non-oscillatory or monotone) reconstructions, the introduction of Discontinuous Galerkin type techniques, the study of *a posteriori* error estimates and adaptive grids. It is worth to emphasize that, at least at a technical level, the ideas of this paper do not allow for a straightforward adaptation to such complex situations.

In the SL scheme, (1.2) is discretized as

$$v_i^{n+1} = I[V^n](x_i - a\Delta t), \tag{2.1}$$

where v_i^{n+1} is an approximation of $v(x_i, t^{n+1})$ and the interpolation $I[V^n](x)$ is computed as

$$I[V^n](x) = \sum_j v_j^n \psi_j(x) \tag{2.2}$$