Journal of Computational Mathematics
Vol.28, No.4, 2010, 552–568.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m0016

A STOPPING CRITERION FOR HIGHER-ORDER SWEEPING SCHEMES FOR STATIC HAMILTON-JACOBI EQUATIONS*

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Abstract

We propose an effective stopping criterion for higher-order fast sweeping schemes for static Hamilton-Jacobi equations based on ratios of three consecutive iterations. To design the new stopping criterion we analyze the convergence of the first-order Lax-Friedrichs sweeping scheme by using the theory of nonlinear iteration. In addition, we propose a fifth-order Weighted PowerENO sweeping scheme for static Hamilton-Jacobi equations with convex Hamiltonians and present numerical examples that validate the effectiveness of the new stopping criterion.

Mathematics subject classification: 65N06, 65N12, 35F21
Key words: Fast sweeping methods, Gauss-Seidel iteration, High order accuracy, Static Hamilton-Jacobi equations, Eikonal equations.

1. Introduction

Consider the following static Hamilton-Jacobi (H-J) equation:

$$\begin{cases} H(\nabla\phi(x), x) = f(x), & x \in \Omega \setminus \Gamma, \\ \phi(x) = g(x), & x \in \Gamma \subset \Omega, \end{cases} \tag{1.1}$$

where $g(x)$ is a positive, Lipschitz continuous function, Ω is an open, bounded polygonal domain in R^d, and Γ is a subset of Ω. $H(p, x)$ is Lipschitz continuous in both arguments, and it is convex and homogeneous of degree one in the first argument.

This class of first-order nonlinear PDEs arise in many applications such as optimal control, differential games, computer vision, geometric optics, and geophysical applications. Thus it is essential to develop efficient high-order accurate numerical methods for such equations. Based on [22, 27] we propose a fifth-order sweeping scheme for the equation. To design an effective stopping criterion for the sweeping scheme, we analyze convergence of the first-order Lax-Friedrichs scheme in terms of theory of nonlinear iterative methods.

Fast sweeping methods are a family of efficient methods for solving static Hamilton-Jacobi equations [3, 7–9, 11, 18, 19, 25, 28, 29], and some essential ideas of these methods may trace

* Received November 29, 2008 / Revised version received May 3, 2009 / Accepted October 17, 2009 / Published online April 19, 2010 /

back to [2, 20]. In [28] the fast sweeping method was systematically analyzed for eikonal equations. Since then the fast sweeping methods have undergone intensive development for general static Hamilton-Jacobi equations in [3, 8, 9, 11, 18, 19, 25, 28, 29] and have found many different applications; see [10] for example. On the other hand, the fast marching method and its relatives consist of another family of numerical methods for solving static Hamilton-Jacobi equations [5, 23, 24, 26].

A fast sweeping method consists of the following three essential ingredients: 1) an efficient local solver for a Hamilton-Jacobi equation on a given Cartesian mesh or triangulation, 2) systematic orderings of solution nodes according to some pre-determined information-flowing directions, and 3) Gauss-Seidel type iterations based on a given order of solution nodes. However, among all the above cited works most of the methods are of first-order accuracy, and only [11, 27] consider higher-order sweeping schemes for such equations. In [27] a third-order WENO scheme [6] is incorporated into Godunov and Lax-Friedrichs numerical Hamiltonians. In [11] a second-order discontinuous-Galerkin discretization is used to design a fast sweeping method for eikonal equations. Here we propose a fifth-order accurate sweeping method in terms of the Weighted PowerENO reconstruction procedure for H-J equations [22] in the same way as proposed in [27] for third-order accurate sweeping methods. The fifth-order fast sweeping method is able to approximate up to high accuracy the solution of multidimensional H-J equations.

Since Gauss-Seidel iteration requires of a criterion to stop the iterative procedure, an absolute stopping criterion to determine the convergence was proposed in [28] such that the algorithm stops when the L^1-norm of the difference between two consecutive iterates is smaller than a given small number. This stopping criterion behaves consistently for first-order fast sweeping methods based on monotone Godunov and Lax-Friedrichs Hamiltonians [8, 28]. However, high-order versions of these methods do not inherit monotonicity and therefore the convergence may be oscillatory. In this case, an absolute stopping criterion is not robust enough to determine the optimal iterate to which the scheme converges.

To design an effective stopping criterion we analyze the convergence of first-order Lax-Friedrichs sweeping methods based on classical results of nonlinear functional analysis [4, 13]. The convergence of the nonlinear Jacobi or Gauss-Seidel iteration resulting from the Lax-Friedrichs sweeping can be proved by using the Banach fixed-point theorem through the explicit expression of the Jacobian matrix. Based on this convergence analysis, we then propose a consistent relative stopping criterion to determine the converged solution of the iterative procedure. We present a series of numerical experiments to validate the proposed new higher order sweeping scheme.

The paper is organized as follows. In Section 2 we review fast sweeping methods and propose a high-order sweeping scheme based on the fifth-order weighted PowerENO scheme. In Section 3 we analyze the convergence of Lax-Friedrichs fast sweeping methods and propose a stopping criterion consistent with the convergence analysis. In Section 4 we present numerical experiments to demonstrate higher order accuracy of the proposed scheme.

2. Fast Sweeping Methods

2.1. A generic formulation for sweeping

We consider a rectangular $n \times n$ mesh, Ω_h, where $x_i = ih_x$ $(i = 1, .., n)$, and $y_j = jh_y$ $(j = 1, ..., n)$ are the grid points. We discretize the nonlinear H-J equation (1.1) by a monotone