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ON EQUILIBRIUM PRICING AS CONVEX OPTIMIZATION*

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Abstract

We study competitive economy equilibrium computation. We show that, for the first time, the equilibrium sets of the following two markets: 1. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions; 2. The Fisher and Arrow-Debreu markets with several classes of concave non-homogeneous utility functions; are convex or log-convex. Furthermore, an equilibrium can be computed as convex optimization by an interior-point algorithm in polynomial time.

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1. Introduction

The study of competitive economy equilibria occupies a central place in mathematical economics. This study was formally started by Walras [14] over a hundred years ago. In this problem everyone in a population of n agents has an initial endowment of divisible goods and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such an equilibrium would exist, under very mild conditions, if the utility functions of agents were concave. Their proof was non-constructive and did not offer any algorithm to compute an equilibrium.

Fisher considered a related and different market model where agents were divided into two sets: producers and consumers; see Brainard and Scarf [2,13]. Consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all budgets are spent and all goods are sold. Fisher's model is a special case of Walras' model when money is also considered a good so that Arrow and Debreu's result applies.

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In a remarkable piece of work, Eisenberg and Gale [8, 11] give a convex programming (or optimization) formulation whose solution yields equilibrium allocations for the Fisher market with linear utility functions, and Eisenberg [9], published 1961 in *Management Science*, extended this approach to derive a convex program for general concave and homogeneous utility functions of degree 1. Their program consists of maximizing an aggregate utility function of all consumers over a convex polyhedron defined by supply-demand linear constraints. The Lagrange or dual multipliers of these constraints yield equilibrium prices. Thus, finding a Fisher equilibrium becomes solving a convex optimization problem, and it could be computed by the Ellipsoid method or by efficient interior-point methods in polynomial time. Later, Codenotti et al. [5] rediscovered the convex programming formulation, and Jain et al. [12] generalized Eisenberg and Gale's convex model to handling homothetic and quasi-concave utilities introduced by Friedman [10]. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$; or, if there is a rational equilibrium solution, one can compute an *exact* equilibrium in a number of arithmetic operations bounded by polynomial in n and L, where L is the bit-length of the input data. When the utility functions are linear, the current best arithmetic operations complexity bound is $\mathcal{O}(\sqrt{mn}(m+n)^3L)$ given by [15].

Little is known on the computational complexity of computing market equilibria for non-homogeneous utility functions or for markets other than the Fisher and Arrow-Debreu settings such as utility functions with externality. This paper is to derive convex programs to solve a couple of more general equilibrium problems. We show that, for the first time, the equilibrium of either of the following two markets:

1. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions;
2. The Fisher and Arrow-Debreu markets with several classes of concave non-homogeneous utility functions;

can be computed as convex optimization and by interior-point algorithms in polynomial time. These markets have wide applications in supply chain and communication spectrum management.

First, a few mathematical notations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space; $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ where each coordinate is non-negative. $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real numbers and the set of non-negative real numbers, respectively.

A function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ is said to be *concave* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$ and any $0 \le \alpha \le 1$, we have

$$u(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \ge \alpha u(\boldsymbol{x}) + (1-\alpha)u(\boldsymbol{y}).$$

It is *homothetic* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$ and any $\alpha > 0$,

$$u(\boldsymbol{x}) \ge u(\boldsymbol{y}) \mathrm{iff} u(\alpha\boldsymbol{x}) \ge u(\alpha\boldsymbol{y}).$$

It is *monotone increasing* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$, $\boldsymbol{x} \ge \boldsymbol{y}$ implies that $u(\boldsymbol{x}) \ge u(\boldsymbol{y})$. It is *homogeneous* of degree d if for any $\boldsymbol{x} \in \mathbf{R}^n_+$ and any $\alpha > 0$, $u(\alpha\boldsymbol{x}) = \alpha^d u(\boldsymbol{x})$.

2. The Fisher and Arrow-Debreu Markets

Without loss of generality, assume that there is one unit of good for each type of good $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) in the Fisher market have an initial money