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WEAK APPROXIMATION OF OBLIQUELY REFLECTED DIFFUSIONS IN TIME-DEPENDENT DOMAINS*

Kaj Nyström and Thomas Önskog
Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden
Email: kaj.nystrom@math.umu.se thomas.onskog@math.umu.se

Abstract

In an earlier paper, we proved the existence of solutions to the Skorohod problem with oblique reflection in time-dependent domains and, subsequently, applied this result to the problem of constructing solutions, in time-dependent domains, to stochastic differential equations with oblique reflection. In this paper we use these results to construct weak approximations of solutions to stochastic differential equations with oblique reflection, in time-dependent domains in $\mathbb{R}^d$, by means of a projected Euler scheme. We prove that the constructed method has, as is the case for normal reflection and time-independent domains, an order of convergence equal to $1/2$ and we evaluate the method empirically by means of two numerical examples. Furthermore, using a well-known extension of the Feynman-Kac formula, to stochastic differential equations with reflection, our method gives, in addition, a Monte Carlo method for solving second order parabolic partial differential equations with Robin boundary conditions in time-dependent domains.

Mathematics subject classification: 65MXX, 35K20, 65CXX, 60J50, 60J60

Key words: Stochastic differential equations, Oblique reflection, Robin boundary conditions, Skorohod problem, Time-dependent domain, Weak approximation, Monte Carlo method, Parabolic partial differential equations, Projected Euler scheme.

1. Introduction

The use of projected Euler schemes, in the construction of weak approximations of solutions to stochastic differential equations with reflection, originates in the work of Saisho [1], who used this approach to prove existence and uniqueness of solutions to stochastic differential equations with normal reflection in time-independent domains. The ideas in [1] were developed further by Costantini, Pacchiarotti and Sartoretto [2], who presented a projected Euler scheme based on the previous work by Costantini [3] concerning the existence of solutions to the Skorohod problem with oblique reflection in time-independent domains. The algorithm proposed in [2] provides, in particular, a Monte Carlo method for solving second order parabolic partial differential equations with mixed Dirichlet and Robin boundary conditions in fairly general time-independent domains. However, in [2] it is proved that the order of convergence of the proposed algorithm is merely $1/2$. In recent years several attempts have been made to find more efficient algorithms for stochastic differential equations, with reflection, but the attempts have only been successful for quite limited sets of boundary conditions. In this context we mention, in particular, the projected Euler schemes suggested by Gobet [4] and by Bossy, Gobet and Talay [5]. The order of convergence of the algorithm proposed in [4], which is based on the

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explicit solution to the Skorohod problem in half-spaces, is 1 in the special case of Neumann boundary conditions with reflection in the conormal direction. On the other hand, the order of convergence of the algorithm in [5] is 1 for all possible directions of reflection, but for this algorithm only a very restricted set of boundary data is allowed. An alternative approach to the problem of weak approximation of solutions to stochastic differential equations with oblique reflection, in time-independent domains with smooth boundary, was given by Mil'shtein [6], who presented two numerical algorithms for second order parabolic partial differential equations with Robin boundary conditions. Although the order of convergence of the fastest of these two algorithms is 1, both algorithms have proved to be difficult to implement due to the fact that a change of coordinate system is required at all time steps at which the approximate solution to the stochastic differential equation is close to the boundary.

An important novelty of the article at hand is that we present an algorithm for weak approximation of stochastic differential equations with oblique reflection in the setting of time-dependent domains and to our knowledge this is indeed an area which is less developed compared to the corresponding problem in time-independent domains. Nevertheless, stochastic differential equations with reflection in time-dependent domains emerge in a variety of applications such as singular stochastic control problems and particle dispersion in volumes with fluctuating size and shape. From a purely theoretical point of view, the study of stochastic differential equations with reflection in time-dependent domains was commenced by Costantini, Gobet and El Karoui [7], who proved existence and uniqueness of solutions to the Skorohod problem with normal reflection in smooth time-dependent domains. Concerning the Skorohod problem we also note that existence and uniqueness for deterministic problems of Skorohod type, in time-dependent intervals, have recently been established by Burdzy et al. (see [8, 9]). In [10] we conducted a thorough study of the multi-dimensional Skorohod problem in time-dependent domains and we proved, in particular, the existence of cádlág solutions to the Skorohod problem, with oblique reflection, in fairly general time-dependent domains. Furthermore, in [10] we, subsequently, used our results on the Skorohod problem to construct solutions to stochastic differential equations with oblique reflection in time-dependent domains. Moreover, in the process of proving these results, we established a number of estimates for solutions, with bounded jumps, to the Skorohod problem. In this article we build on the study in [10] and we use the results in [10] regarding the Skorohod problem to develop an algorithm for weak approximation of stochastic differential equations with oblique reflection in the setting of time-dependent domains. Our approximation procedure is, from a numerical point of view, similar to the projected Euler scheme described in [2], but our setting is different and more general compared to [2] as we, in particular, allow for time-dependent domains, more general functionals as well as reflection in oblique directions. By proceeding along the lines of [2] we also prove that the proposed algorithm has an order of convergence equal to 1/2 and we emphasize that while this convergence may seem slow, the main advantage of the approach is, and this makes it different from the algorithms and results in [4–6], that the method outlined is applicable in very general situations.

To briefly outline the general result, established in [10], concerning stochastic differential equations with oblique reflection in time-dependent domains, and to formulate the results of this article, we next introduce some notation. Given $d \geq 1$, we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$ and we let $|z| = \langle z, z \rangle^{1/2}$ be the Euclidean norm of z. Whenever $z \in \mathbb{R}^d$, $r > 0$, we let

$$B_r(z) = \left\{ y \in \mathbb{R}^d : |z - y| < r \right\} \quad \text{and} \quad S_r(z) = \left\{ y \in \mathbb{R}^d : |z - y| = r \right\}.$$