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THE REDUCED BASIS TECHNIQUE AS A COARSE SOLVER FOR PARAREAL IN TIME SIMULATIONS*

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Abstract

In this paper, we extend the reduced basis methods for parameter dependent problems to the parareal in time algorithm introduced by Lions *et al.* [12] and solve a nonlinear evolutionary parabolic partial differential equation. The fine solver is based on the finite element method or spectral element method in space and a semi-implicit Runge-Kutta scheme in time. The coarse solver is based on a semi-implicit scheme in time and the reduced basis approximation in space. Offline-online procedures are developed, and it is proved that the computational complexity of the on-line stage depends only on the dimension of the reduced basis space (typically small). Parareal in time algorithms based on a multi-grids finite element method and a multi-degrees finite element method are also presented. Some numerical results are reported.

Mathematics subject classification: 52B10, 65D18, 68U05, 68U07.
Key words: Finite element and spectral element approximations, Multi-meshes and multi-degrees techniques, Reduced basis technique, Semi-implicit Runge-Kutta scheme, Offline-online procedure, Parareal in time algorithm.

1. Introduction

The parareal in time algorithm allows to use parallel computers for the approximation of the solution to ordinary or evolution partial differential equations by decomposing the time integration interval into time slabs and iterating on the resolution over each time slab to converge to the global solution. The iterations combine in a predictor/corrector way the use of a coarse propagator that is inexpensive and a precise solver (that is used only in parallel over each time slab, allocated to different processors); see, e.g., [2,3,17,18]. In many instances the iterative schemes provide an approximate solution as accurate as if the precise solver would be used over the complete time integration interval. One of the expensive parts of the solver is the resolution of the coarse solver since it is used sequentially over the complete time integration interval. Our goal is the development of numerical methods that permit the efficient evaluation of parareal in time simulation.

To achieve this goal we will pursue the reduced basis method. The reduced basis method was first introduced in the late 1970s for the nonlinear analysis of structures [1,19,20] and has subsequently been further investigated and developed more broadly; see, e.g., [4,5,9,21,22,24]. In the more recent past the reduced basis approach and in particular associated a posteriori error estimation procedures have been successfully developed for the PDEs with affine parameter or time dependence; see, e.g., [10,15,16,23,26]. Indeed, the reduced basis technique allows, from

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a few preliminary computations with a standard solver, to generate basis functions adapted to the further approximation of problems that depend on a parameter. This is a very high order approximation method, in the sense where, when the set of all solutions to the parameter dependent problem has a small width and with (much) less than 100 degrees of freedom, a very good approximation is available (the accuracy is about the one obtained with discretization). In more general cases where the dependency of the solutions in the parameter is not so regular, the number of degrees of freedom may become too large to get an acceptable accuracy. In this paper, we consider the extension of the reduced basis method to define a coarse and very cheap propagator that allows to get the full efficiency in a parareal context.

Many numerical methods are considered to define the coarse propogator in the literature [17]: the most well-known ones are the usual coarse mesh of the finite element method (FEM), the spectral approximation space based on the polynomial of lower degree, and a coarser model based on simpler physics. The success of these numerous experiments not only richen the idea of parareal in time algorithm, but also motivate the need for further studies in this direction. The main contributions here are as follows: (i) we construct a coarse propagator based on a semi-implicit scheme in time and the reduced basis approximation in space, and prove that the computational complexity of the on-line stage of the procedure scales only with the dimension of the reduced basis space (this fact means that good accuracy is obtained even for very few basis functions, and thus the computational cost of the coarse solver is typically very small); (ii) we propose a fine propagator based on the FEM or spectral element method (SEM) in space and a semi-implicit Runge-Kutta (RK) scheme in time; (iii) the parareal in time algorithm based on a multi-degrees FEM in space and the semi-implicit RK scheme in time is considered.

This paper is organized as follows: Section 2 describes the basic algorithm for a model equation. Section 3 introduces the necessary notations and the initial-boundary problem which is considered in this paper and proposes two types of fine approximated propagators based on the FEM and SEM in space and semi-implicit RK scheme in time. Section 4 introduces the coarse approximated propagator based on the reduced basis method. Section 5 gives the parareal in time algorithms based on the multi-grids FEM and the multi-degrees FEM. Some numerical results are reported in Section 6, and finally we give some conclusions in Section 7.

2. Basic Algorithm on a Model Equation

Consider the following time dependent problem

$$\frac{\partial u}{\partial t} + Lu = 0, \;\; u(0) = u^0, \tag{2.1}$$

where, for the sake of simplicity, the operator L does not depend on time. We introduce the propagator S such that $S_\tau(v)$ is the solution, at time τ of the problem

$$\frac{\partial u}{\partial t} + Lu = 0, \;\; u(0) = v. \tag{2.2}$$

Due to time invariance, it is well-known that

$$S_\tau = S_{\tau-t} \circ S_t, \qquad \forall t < \tau. \tag{2.3}$$

Let $0 = T_0 < T_1 < \cdots < T_n < \cdots < T_M = T$ be special times at which we are interested to consider snapshots of the solution $u(T_n)$. Then we obtain from (2.2) and (2.3) that

$$u(T_{n+1}) = S_{T_{n+1}}(u^0) = S_{T_{n+1}-T_n}(u_{T_n}).$$