Journal of Computational Mathematics
Vol.28, No.6, 2010, 725–744.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m0004

ERROR ESTIMATES FOR THE RECURSIVE LINEARIZATION OF INVERSE MEDIUM PROBLEMS*

Gang Bao
Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027,USA
Email: bao@math.msu.edu
Faouzi Triki
LMC-IMAG, Université Joseph Fourier I, B.P. 53, 38041 Grenoble Cedex 9, France
Email: Faouzi.Triki@imag.fr

Abstract

This paper is devoted to the mathematical analysis of a general recursive linearization algorithm for solving inverse medium problems with multi-frequency measurements. Under some reasonable assumptions, it is shown that the algorithm is convergent with error estimates. The work is motivated by our effort to analyze recent significant numerical results for solving inverse medium problems. Based on the uncertainty principle, the recursive linearization allows the nonlinear inverse problems to be reduced to a set of linear problems and be solved recursively in a proper order according to the measurements. As an application, the convergence of the recursive linearization algorithm [Chen, Inverse Problems 13(1997), pp.253-282] is established for solving the acoustic inverse scattering problem.

Mathematics subject classification: 35R30, 65N30, 78A46
Key words: Recursive linearization, Tikhonov regularization, Inverse problems, Convergence analysis.

1. Introduction

Motivated by significant scientific and industrial applications, the field of inverse problems has undergone a tremendous growth in the last several decades. A variety of inverse problems, including identification of PDE coefficients, reconstruction of initial data, estimation of source functions, and detection of interfaces or boundary conditions, demand the solution of ill-posed non-linear operator equations see, e.g., [12, 18]. Our focus of this paper is on the inverse medium scattering problem, *i.e.*, the reconstruction of the refractive index of an inhomogeneous medium from measurements of the far field pattern of the scattered fields. The inverse medium scattering problem arises naturally in diverse applications such as radar, sonar, geophysical exploration, medical imaging, and nondestructive testing. There are two major difficulties associated with the nonlinear inverse problem: the ill-posedness and the presence of many local minima. A number of algorithms have been proposed for numerical solutions of this inverse problem. Classical iterative optimization methods offer fast local convergence but often fail to compute the global minimizers because of multiple local minima. Another main difficulty is the ill-posedness, *i.e.*, infinitesimal noise in the measured data may give rise to a large error in the computed solution. It is well known that the ill-posedness of the inverse scattering problem decreases as the frequency increases. However, at high frequencies, the nonlinear equation

* Received May 25, 2009 / Revised version received September 10, 2009 / Accepted October 26, 2009 / Published online August 9, 2010 /

becomes extremely oscillatory and possesses many more local minima. A challenge for solving the inverse problem is to develop solution methods that take advantages of the regularity of the problem for high frequencies without being undermined by local minima.

To overcome the difficulties, stable and efficient regularized recursive linearization methods are developed in [3, 9, 10] for solving the two-dimensional Helmholtz equation and the three-dimensional Maxwell's equations [4] in the case of full aperture data. We refer the reader to [3, 5, 6] for limited aperture data cases. Roughly speaking, these methods use the Born approximation at the lowest frequency k_{min} to obtain the initial guesses which are the low-frequency modes of the medium. Updates are made by using the data at higher frequency sequentially until a sufficiently high frequency k_{max} where the dominant modes of the medium are essentially recovered.

In the case of fixed frequencies, a related continuation approach has been developed on the spatial frequencies [3]. A recursive linearization approach has also been developed in [11] for solving inverse obstacle problems. More recently, direct imaging techniques have been explored to replace the weak scattering for generating the initial guess [2]. Although the numerical results are efficient and robust, the analysis of the computational methods is completely open. Our main goal of this paper is to originate the convergence analysis of the general recursive linearization algorithm for solving the inverse medium problem. Under some reasonable assumptions, we establish the convergence of the algorithm along with an error estimate. Our analysis is inspired by the underlying physics, especially the uncertainty principle.

The outline of the paper is as follows. A formulation of the nonlinear inverse scattering problem is presented in Section 2. Section 3 is devoted to useful properties of the linearized problem. In Section 4, we discuss the significance of the uncertainty principle in the study of inverse problems. Through a singular value decomposition analysis, the uncertainty principle may further be used to characterize the ill-posedness of the inverse problem. A reconstruction method based on the uncertainty principle, recursive linearization, is introduced. We establish the convergence of the recursive linearization approach and derive an error estimate in Section 5. As an example, we apply the convergence result to the algorithm presented in [9] for solving an inverse medium scattering problem in Section 6. Finally, some relevant discussions are provided in the Appendix about the uncertainty principal and its close connection to the inverse medium scattering problem.

2. Inverse Medium Scattering Problem

The scattering of time-harmonic electromagnetic waves by a cylindrical shaped inhomogeneous medium with refractive index $1 + q(x)$ is governed by the following differential equation

$$\Delta\phi(x) + k^2(1 + q(x))\phi(x) = 0, \text{in} \mathbb{R}^2, \tag{2.1}$$

where the real part of the complex valued function ϕ describes the space-dependent part of a velocity potential in the case of acoustic waves or an electric/magnetic field in the case of electromagnetic waves. The real number $k > 0$ is the wave number. Assume that the refractive index $q(x) + 1$ is a positive real function in $\mathbb{R}^2$, the scatterer $q(x)$ is compactly supported in $D(R)$ and belongs to $C_0^2(D)$. Here $D(R)$ denotes a ball in $\mathbb{R}^2$ centered at 0 with radius R. The direct or forward scattering problem in this context is for a given incident wave $\phi_0(x)$ satisfying the Helmholtz equation $\Delta\phi_0 + k^2\phi_0 = 0$ in $\mathbb{R}^2$, to determine the scattered wave $\psi(x) : \mathbb{R}^2 \to \mathbb{C}$