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BLOCK-TRIANGULAR PRECONDITIONERS FOR SYSTEMS ARISING FROM EDGE-PRESERVING IMAGE RESTORATION*

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Abstract

Signal and image restoration problems are often solved by minimizing a cost function consisting of an ℓ_2 data-fidelity term and a regularization term. We consider a class of convex and edge-preserving regularization functions. In specific, half-quadratic regularization as a fixed-point iteration method is usually employed to solve this problem. The main aim of this paper is to solve the above-described signal and image restoration problems with the half-quadratic regularization technique by making use of the Newton method. At each iteration of the Newton method, the Newton equation is a structured system of linear equations of a symmetric positive definite coefficient matrix, and may be efficiently solved by the preconditioned conjugate gradient method accelerated with the modified block SSOR preconditioner. Our experimental results show that the modified block-SSOR preconditioned conjugate gradient method is feasible and effective for further improving the numerical performance of the half-quadratic regularization approach.

Mathematics subject classification: 65C20, 65F10.
Key words: Block system of equations, Matrix preconditioner, Edge-preserving, Image restoration, Half-quadratic regularization.

1. Introduction

We consider the signal and image restoration problems for which a vector $\hat{\mathbf{x}} \in \mathbb{R}^p$ (an image or a signal) is estimated based upon a degraded data vector $\mathbf{b} \in \mathbb{R}^q$ by minimizing a cost function $J : \mathbb{R}^p \to \mathbb{R}$. The function J is a combination of a data-fidelity term with a regularization term Φ that is weighted by a parameter $\beta > 0$. More precisely, the problems are of the form

$$\hat{\mathbf{x}} = \min_{\mathbf{x}\in\mathbb{R}^p} J(\mathbf{x}),$$
$$J(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \beta\Phi(\mathbf{x}),$$

where $\mathbf{A} \in \mathbb{R}^{q\times p}$ is the known blurring matrix. The data-fidelity term given above assumes that $\mathbf{b}$ and $\mathbf{x}$ satisfy an approximate linear relation $\mathbf{A}\mathbf{x} \approx \mathbf{b}$, but that $\mathbf{b}$ is contaminated by noise.

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Such data-fidelity terms are popular in numerous inverse problems such as seismic imaging, non-destructive evaluation, and x-ray tomography, see for instance [11]. Here, we consider regularization terms Φ of the form

$$\Phi(\mathbf{x}) = \sum_{i=1}^{r} \phi(\mathbf{g}_i^T \mathbf{x}), \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and $\mathbf{g}_i : \mathbb{R}^p \to \mathbb{R}$, for $i = 1, \cdots, r$, are linear operators. Typically, $\{\mathbf{g}_i^T \mathbf{x}\}$ are the first- or the second-order differences between the neighboring samples in $\mathbf{x}$. Let $\mathbf{G}$ denote the $r \times p$ matrix whose ith row is $\mathbf{g}_i^T$, for $i = 1, \cdots, r$, and assume that

$$\mathbf{A} \not\equiv 0, \quad \mathbf{G} \not\equiv 0, \quad \phi \not\equiv 0 \quad \text{and} \quad \ker(\mathbf{A}^T\mathbf{A}) \cap \ker(\mathbf{G}^T\mathbf{G}) = \{0\}, \tag{1.2}$$

where $\ker(\cdot)$ denotes the kernel of the corresponding matrix. Clearly, this assumption guarantees that $\alpha_1 \mathbf{A}^T\mathbf{A} + \alpha_2 \mathbf{G}^T\mathbf{G}$ is a symmetric positive definite matrix provided both α_1 and α_2 are positive constants.

In this paper, we focus on convex, *edge-preserving* potential functions $\phi : \mathbb{R} \to \mathbb{R}$ defined in (1.1), because they can yield image and signal estimates of high quality, involving edges and homogeneous regions. Typical examples of such functions are:

$$\phi_1(t) = |t|/\alpha - \log(1 + |t|/\alpha), \tag{1.3}$$

$$\phi_2(t) = \sqrt{\alpha + t^2}, \tag{1.4}$$

$$\phi_3(t) = \log(\cosh(\alpha t))/\alpha, \tag{1.5}$$

$$\phi_4(t) = \begin{cases} t^2/(2\alpha), & \text{if} \quad |t| \le \alpha, \\ |t| - \alpha/2, & \text{if} \quad |t| > \alpha, \end{cases} \tag{1.6}$$

where $\alpha > 0$ is a prescribed parameter. See [15] and the references therein. We will consider the case that ϕ is convex, even, and is $\mathbb{C}^2$, and that

$$\mathbf{A}^T\mathbf{A} \text{ is invertible} \quad \text{and/or} \quad \phi\text{"}(t) > 0, \quad \forall t \in \mathbb{R}. \tag{1.7}$$

The assumptions in (1.7) and (1.2) guarantee that for every $\mathbf{x} \in \mathbb{R}^p$, the function J has a unique minimum and that this minimum is strict. As now $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite, we know that $\ker(\mathbf{A}^T\mathbf{A}) \cap \ker(\mathbf{G}^T\mathbf{G}) = \{0\}$ holds true. Moreover, we easily see that $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite if and only if $q \ge p$ and $\mathbf{A}$ is of full column rank.

However, the minimizers $\hat{\mathbf{x}}$ of the cost-functions J involving edge-preserving regularization are nonlinear with respect to $\mathbf{x}$ and their computations are quite costly. To simplify the computation, a half-quadratic reformulation of J was pioneered in [12] and [13]. One of the basic idea is to construct an augmented cost function $\tilde{J} : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}$ that involves an auxiliary variable $\mathbf{z} \in \mathbb{R}^r$ in the following form:

$$\tilde{J}(\mathbf{x}, \mathbf{z}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \beta \sum_{i=1}^{r} \left(\frac{1}{2} (\mathbf{g}_i^T \mathbf{x} - z_i)^2 + \psi(z_i) \right), \tag{1.8}$$

where

$$\psi(t) = \min_{s \in \mathbb{R}} \left\{ -\frac{1}{2}(t - s)^2 + \phi(s) \right\}, \quad \forall t \in \mathbb{R}, \tag{1.9}$$