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ON CONTRACTION AND SEMI-CONTRACTION FACTORS OF GSOR METHOD FOR AUGMENTED LINEAR SYSTEMS*

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Abstract

The *generalized successive overrelaxation* (**GSOR**) method was presented and studied by Bai, Parlett and Wang [Numer. Math. 102(2005), pp.1-38] for solving the augmented system of linear equations, and the optimal iteration parameters and the corresponding optimal convergence factor were exactly obtained. In this paper, we further estimate the contraction and the semi-contraction factors of the GSOR method. The motivation of the study is that the convergence speed of an iteration method is actually decided by the contraction factor but not by the spectral radius in finite-step iteration computations. For the nonsingular augmented linear system, under some restrictions we obtain the contraction domain of the parameters involved, which guarantees that the contraction factor of the GSOR method is less than one. For the singular but consistent augmented linear system, we also obtain the semi-contraction domain of the parameters in a similar fashion. Finally, we use two numerical examples to verify the theoretical results and the effectiveness of the GSOR method.

Mathematics subject classification: 65F10, 65F50; CR: G1.3.
Key words: Contraction and semi-contraction factors, Augmented linear system, GSOR method, Convergence.

1. Introduction

We study an iterative solution of the augmented linear system

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \quad or \quad \mathcal{A}z = f, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite, and $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix. Here $m \geq n$, and $b \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ are given vectors, and $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are unknown vectors, respectively. We use B^T to denote the transpose of the matrix B. When B is of full column-rank, we know that the augmented linear system (1.1) has a unique solution. When

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B is rank-deficient and $q \in R(B^T)$ (the range of B^T), the augmented linear system (1.1) has infinitely many solutions; which is called the singular but consistent augmented linear system.

The augmented linear system (1.1) results from a wide variety of scientific and engineering applications such as mixed and hybrid finite element approximations of the elliptic problems, Stokes equations, weighted least-squares problems, computer graphics, electronic networks and others; see [1, 2]. The augmented linear system is also called as a saddle point problem, or a *Karush-Kuhn-Tucker* (**KKT**) system. Recently, the augmented linear system has attracted more and more researchers and various kinds of iteration methods have been established and discussed. For example, the Uzawa-type methods [13, 15], the preconditioned Krylov subspace methods [7, 9, 18], the relaxation methods [8, 10, 14, 17], and the Hermitian and skew-Hermitian splitting methods [3–6, 12], etc. Moreover, the singular augmented linear system has been specially studied in [11, 19].

The oldest and famous iteration method for solving the augmented linear system is the Uzawa method [1]. Gloub et al. proposed an SOR-like method for solving the linear system (1.1) in [16]. Based on this idea, Bai et al. established and discussed the GSOR method in [8] and obtained the optimal parameters and the corresponding optimal convergence factor; see also [10].

The GSOR method has the following form.

Method 1.1. ([8]) (*The GSOR Method*).
Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric and nonsingular matrix. Given initial vectors $x^{(0)} \in \mathbb{R}^m$ and $y^{(0)} \in \mathbb{R}^n$, and two relaxation factors ω, τ with $\omega, \tau \neq 0$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{({x^{(k)}}^T, {y^{(k)}}^T)^T\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(b - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^T x^{(k+1)} + q). \end{cases}$$

Here, Q is an approximate (preconditioning) matrix of the Schur complement matrix $B^T A^{-1} B$.

We know that an iteration method is convergent when the spectral radius of the corresponding iteration matrix is less than one. As a matter of fact, the convergence speed of an iteration method is, however, decided by the contraction factor, but not by the spectral radius in practical computations. Therefore, to estimate the contraction factor of an iteration method is a practically important task.

In this paper, firstly, we give the iteration matrix of the GSOR method and introduce a new norm. According to this norm, we propose the concept about the contraction factor of the GSOR method. Usually, it is difficult to obtain the optimal parameters which minimize the contraction factor. Hence, we turn to estimate an upper bound of the contraction factor proposed. The domain makes the upper bound be less than one. Moreover, we extend these results to the singular but consistent augmented linear system.

The paper is organized as follows. In Section 2, the contraction and semi-contraction factors of the GSOR method are established, and the convergence and the semi-convergence of the GSOR method are analyzed. In Section 3, the domains of parameters which guarantee the contraction and the semi-contraction factors to be less than one are obtained. Numerical examples are given in Section 4.