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COMPUTATIONAL ISSUES IN SENSITIVITY ANALYSIS FOR 1-D INTERFACE PROBLEMS*

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Abstract

This paper is concerned with the construction of accurate and efficient computational algorithms for the numerical approximation of sensitivities with respect to a parameter dependent interface location. Motivated by sensitivity analysis with respect to piezoelectric actuator placement on an Euler-Bernoulli beam, this work illustrates the key concepts related to sensitivity equation formulation for interface problems where the parameter of interest determines the location of the interface. A fourth order model problem is considered, and a homogenization procedure for sensitivity computation is constructed using standard finite element methods. Numerical results show that proper formulation and approximation of the sensitivity interface conditions is critical to obtaining convergent numerical sensitivity approximations. A second order elliptic interface model problem is also mentioned, and the homogenization procedure is outlined briefly for this model.

Mathematics subject classification: 65N06, 65B99.
Key words: Finite element method, Interface Problems Sensitivity Equation.

1. Introduction

Scientists often want to measure how well a mathematical model represents the fundamental behaviors of a physical system, and they are often charged with the task of quantifying some measure of how the uncertainty in the model parameters proliferates into uncertainty in the results of the model simulations. As pointed out in [1], sensitivity analysis and uncertainty analysis combine to produce a systematic approach to developing a comprehensive understanding of a mathematical model, the data it produces, and the way that the data is used to influence the design of many engineering systems. Accurate sensitivity calculations play an important role in this process. The term *Sensitivity Equation Methods* (SEMs) refers to a large class of techniques that attempt to derive, analyze, and solve equations whose solutions are functions referred to as *sensitivities*. Sensitivities are derivatives which describe how small changes in design parameters affect the state variables of a mathematical model. Continuous Sensitivity Equation Methods (CSEMs) are one such technique in this class of methods. CSEMs have been used to compute gradients and greatly improve design cycle times in optimization-based design, see [2–4]. In

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addition, they can be used to construct fast solvers for computational fluid dynamics [5] and are essential to quantifying uncertainties in parameter dependent systems [1, 6].

The CSEM approach requires one to first derive the appropriate sensitivity equation, then to show the resulting equation is well posed in an appropriate function space, and finally to develop good numerical schemes for approximating the sensitivities. In certain situations, such as when geometry or shape parameters are considered, the sensitivity equations may have very weak solutions (e.g., only L^2 in space) and require that one develop numerical algorithms that capture these weak solutions, see [7] for an example of this process.

A valid question to ask is whether the development of special numerical methods for approximating the sensitivities is necessary; can simple, "natural," computational methods be used to obtain reasonable sensitivity approximations? Furthermore, is it also essential to analyze the continuous sensitivity equation to show that it is a properly posed mathematical problem? In this work, we show that, in general, the answer to both of these questions is yes. Specifically, we consider a fourth order elliptic problem where the parameter of interest governs the location of a coefficient discontinuity. We summarize our results as follows:

- We give a proper formulation of the continuous sensitivity equation and use this formulation to construct a convergent numerical scheme for approximating the sensitivity.
- We consider a simple, "natural," computational method for approximating the sensitivity and show that it completely fails to yield convergent sensitivity approximations.
- We show that the reason for the failure of this methods is that it fails to recognize a certain property of the sensitivity; furthermore, this property can only be found through a preliminary analysis of the problem.

The main goal of this paper is to illustrate that when applying sensitivity analysis techniques to partial differential equations with discontinuous coefficients, or *interface problems* as they are sometimes called, it is necessary to both analyze the continuous sensitivity equation and to develop a special numerical method to accurately approximate the sensitivity.

The fourth order model problem we consider in this work shares many similarities with the Euler-Bernoulli beam model considered in [8]. Specifically, the model problem contains discontinuous coefficients where the location of the discontinuity serves as the parameter of interest for the sensitivity analysis. Furthermore, the derivation of the sensitivity equation for this simple model exhibits similar issues to that of the original Euler-Bernoulli beam model. However, one can explicitly write down the solution to the model equation and derive an explicit form of the sensitivity variable. This allows us to identify some of the key ideas that are relevant when applying sensitivity analysis to interface problems. We note that our results outlined above for the model problem most certainly apply to the corresponding sensitivity approximations for the more complicated Euler-Bernoulli beam model with patch actuators.

We begin with notation and an outline of the motivating beam model in Section 2. The simplified model problem is presented in Section 3 along with the exact solution of the problem and the corresponding sensitivity. Section 3.1 gives a brief description of the standard finite element formulation for constructing state variable approximations. The continuous sensitivity equation (CSE) is studied in Section 4. A homogenization procedure is used to prove that the CSE is well posed, and a corresponding numerical technique is used to obtain convergent numerical sensitivity approximations. Section 5 uses one type of *Discretize-then-Differentiate* (DD) methodology for deriving a sensitivity equation. Numerical experiments are shown to yield