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ON MATRIX AND DETERMINANT IDENTITIES FOR COMPOSITE FUNCTIONS*

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Abstract

We present some matrix and determinant identities for the divided differences of the composite functions, which generalize the divided difference form of Faà di Bruno's formula. Some recent published identities can be regarded as special cases of our results.

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1. Introduction

The Bell polynomials which are studied extensively by Bell [2] have played a very important role in combinatorial analysis. Many sequences such as the Stirling numbers and the Lah numbers are special values of the Bell polynomials. Recently, Abbas and Bouroubi [1] proposed two new methods for determining new identities for the Bell polynomials, and Yang [20] generalized one of the methods. By studying the matrices related to the Bell polynomials, Wang and Wang [16] gave a unified approach to various lower triangular matrices such as the Stirling matrices of the first kind and of the second kind [5,6], the Lah matrix [14] and so on. The Faà di Bruno formula [7,8] on higher derivatives of composite functions has wide applications in many branches of mathematics, notably in numerical analysis [9,15,17]. The Bell polynomial is one of the representation tools of the Faà di Bruno's formula. Chu [3] used the properties of the Bell polynomials and the Faà di Bruno formula to obtain several classical determinant identities for composite functions which generalized Mina's identities and their extensions due to Kedlaya [11] and Wilf [19].

Divided differences as the coefficients in a Newton form arise in numerical analysis, which also have applications in the study of spline interpolation and polynomial interpolation. Given a function h, for distinct points $x_0, x_1, \cdots, x_n$, the divided differences of h are defined recursively as

$$\begin{aligned} &h[x_0] = h(x_0), \\ &h[x_0, x_1, \cdots, x_n] = \frac{h[x_0, x_1, \cdots, x_{n-1}] - h[x_1, x_2, \cdots, x_n]}{x_0 - x_n}, \qquad n \geq 1. \end{aligned}$$

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By extending the definition for $h[x_0, x_1, \cdots, x_n]$ in the case of distinct arguments, we have a similar formula for $x_0 \le x_1 \le \cdots \le x_n$ as follows

$$h[x_0, x_1, \cdots, x_n] = \begin{cases} \dfrac{h[x_0, x_1, \cdots, x_{n-1}] - h[x_1, x_2, \cdots, x_n]}{x_0 - x_n}, & \text{if } x_n \ne x_0, \\ \dfrac{h^{(n)}(x_0)}{n!}, & \text{if } x_n = x_0. \end{cases}$$

For more basic properties of divided differences, one can refer to a recent reference [4]. In [18], Wang and Wang proposed a divided difference form of Faà di Bruno's formula to obtain an explicit divided difference formula for a composite function.They also provided a new proof of Faà di Bruno's formula. It is also noted that Floater and Lyche [10] independently obtained similar results.

Usually, derivatives can be understood as the limit of divided differences. That is to say, divided differences may be taken as discrete derivatives. This gives a motivation to present some matrix identities and determinant identities for the divided differences of the composite functions. This paper is organized as follows. In Section 2, some notation will be introduced. Section 3 is devoted to the matrix identities for composite functions with divided differences. Using the divided difference form of the Faà di Bruno's formula, Section 4 presents some determinant identities for composite functions.

2. Notation

In this section we will introduce some notation. Let the composite function $h(t) = f(g(t))$. Then the divided difference form of Faà di Bruno's formula [10,18] is described as follows.

Proposition 2.1. *For $n \ge 1$, if f and g are sufficiently smooth functions, then*

$$h[t_0, t_1, \cdots, t_n] = \sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)] \times \sum_{m=\nu_0 \le \nu_1 \le \cdots \le \nu_m = n} \prod_{i=1}^{m} g[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}].$$

In particular, Faà di Bruno's formula holds when $t_0 = t_1 = \cdots = t_n = t$, namely,

$$h^{(n)}(t) = \sum_{m=1}^{n} f^{(m)}(g(t)) B_{n,m}\Big(g'(t), g''(t), \cdots, g^{(n)}(t)\Big),$$

where $B_{n,m}$ is the exponential partial Bell polynomial defined as

$$B_{n,m}(x_1, x_2, \cdots, x_n) = \sum_{\substack{c_1+2c_2+\cdots+nc_n=n \\ c_1+c_2+\cdots+c_n=m}} \frac{n!}{c_1!(1!)^{c_1} c_2!(2!)^{c_2} \cdots c_n!(n!)^{c_n}} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}.$$

Further, we define

$$e_{n,m}(\phi, \{t_i\}_{i=0}^n) = \begin{cases} \displaystyle\sum_{m=\nu_0 \le \nu_1 \le \cdots \le \nu_m = n} \prod_{i=1}^{m} \phi[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}], & 1 \le m \le n, \\ 0, & m > n \text{ or } 0 = m < n, \\ 1, & m = n = 0. \end{cases} \tag{2.1}$$