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HIGH-ORDER LOCAL ABSORBING BOUNDARY CONDITIONS FOR HEAT EQUATION IN UNBOUNDED DOMAINS*

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Abstract

With the development of numerical methods the numerical computations require higher and higher accuracy. This paper is devoted to the high-order local absorbing boundary conditions (ABCs) for heat equation. We proved that the coupled system yields a stable problem between the obtained high-order local ABCs and the partial differential equation in the computational domain. This method has been used widely in wave propagation models only recently. We extend the spirit of the methodology to parabolic ones, which will become a basis to design the local ABCs for a class of nonlinear PDEs. Some numerical tests show that the new treatment is very efficient and tractable.

Mathematics subject classification: 65M06, 65M12, 65M15.
Key words: Heat equation, High-order method, Absorbing boundary conditions, Parabolic problems in unbounded domains.

1. Introduction

Heat equation rises from many fields, for examples, the heat transfer, fluid dynamics, astrophysics, finance or other areas of applied mathematics. In this paper, we consider the numerical solutions of heat equation on unbounded spatial domains. A real challenge is the unboundedness of the physical domains, the traditional methods (finite element method and finite difference method) can not be used in a straight forward manner. Therefore, many mathematicians, engineers and physicists are attracted and devoted to the study of these problems. In the early literatures, Givoli [8] studied the heat problems on unbounded domains, in which the author tried to get DtN artificial boundary condition on the given artificial boundary. Greengard and Lin [11] developed a new algorithm for solving the heat problem on unbounded domains, the algorithm was based on the evolution of the continuous spectrum of the solution. Li and Greengard [25] also proposed a fast solver for heat equation in free space. Strain [29] presented efficient and accurate new adaptive methods related to the fast Gauss transform. Han and Huang [18, 19] presented an exact artificial boundary condition to reduce the original heat equation to an initial-boundary-value problem on a finite computational domain. Wu and Sun [32] constructed a finite difference scheme for one-dimensional case and proved that the scheme was uniquely solvable, unconditionally stable and convergent with the order two in space and the order 3/2 in time under an energy norm. Zheng [41] considered the approximation,

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stability and fast evaluation of 1-D heat equation. Han and Yin [22] presented the numerical solution of 3-D parabolic problems.

The ABCs include the global ABCs and local ABCs. The global ABCs are usually the natural integral equation, i.e., DtN mapping and hence, lead to the well-approximation and well-posed truncated problems, but the implementation cost is expensive. For a great details one can refer to [1, 3, 7, 12, 14, 17, 20, 21, 23, 33, 34, 36, 39, 40, 42] and references therein. On the other hand, local ABCs are computationally efficient, but the accuracy and stability are the main concerns. Enquist and Majda [6] proposed a whole family of local boundary conditions for wave equation, which not only resulted in stable difference approximation, but also minimized the unphysical reflections.

For long time simulation or when the mesh size is small enough, it needs to increase the order of the local ABCs. The high-order method has been used in wave propagation models [5, 27, 28, 30, 31]. Works in [6, 26] suggested the higher-order paraxial approximation as artificial boundary conditions. Based on the use of auxiliary functions, [2] proposed a family of paraxial wave equation approximation, and Collino developed the high-order ABCs for the 2-D wave equation and gave the boundary conditions at corners. Givoli et al [9, 10] proposed a new high-order ABCs for time-dependent wave problems in unbounded domains. More works and their extensions [13, 15, 16], associated with ABCs, improved the work of the Givoli-Neta in some respects (accuracy and stability).

Recently, Zhang, Xu and Wu [37, 38], proposed a novel unified approach to design the local ABCs for nonlinear Schrödinger equation. Based on the well-known operator-splitting method, the procedure of unified approach is to approximate the linear subproblem by distinguishing the incoming and outgoing wave; then unite the resulting approximate operator and the nonlinear subproblem to obtain nonlinear boundary conditions. Brunner, Wu and Zhang [4] successfully applied the method to semilinear parabolic equation on unbounded spatial domain, where the design of local ABCs plays an important role to get the suitable approximate operator for heat equation. In this paper we extend the spirit of the high-order methodology to constructing high-order ABCs for heat equations, and prove that the resulting ABCs are stable. By applying Laplace and Fourier transforms and their inverse transforms, we approximate the one-way equation to obtain the high-order boundary conditions by padé polynomial at expansion point z_0, and introduce the specially defined auxiliary variables to avoid the high derivatives beyond order two, which make the formula tractable when N is chosen larger.

The brief description of this paper is as follows. Section 2 is devoted to the construction of high-order ABCs. In Section 3 the focus of the presentation is on the stability analysis for the reduced initial-boundary-value problems. In Section 4 some numerical examples show the tractability and effectiveness of the high-order ABCs. We end the paper with some concluding remarks.

2. Design of High-Order Absorbing Boundary Conditions

Denote the spatial coordinate by $\mathbf{x}$, which for one-dimensional case is $\mathbf{x} = x$, two-dimensional case is $\mathbf{x} = (x, y)$, and three-dimensional case is $\mathbf{x} = (x, y, z)$. Denote the infinity domain by Ω, the computational domain by Ω_i, the boundary by $\Gamma = \partial\Omega$, and the exterior domain by $\Omega_e = \Omega \setminus \Omega_i$. Heat equation can be written as follows:

$$u_t = a^2 \Delta u + f(\mathbf{x}, t), \qquad \text{in } \Omega,\ t > 0, \tag{2.1}$$