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FINITE ELEMENT METHOD WITH SUPERCONVERGENCE FOR NONLINEAR HAMILTONIAN SYSTEMS*

Chuanmiao Chen
College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China
Email: cmchen@hunnu.edu.cn
Qiong Tang
Department of Information and Computation, Hunan University of Technology, Zhuzhou 412008, China
Email: zzgxysx@163.com
Shufang Hu
College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China
Email: shufang-hu163@163.com

Abstract

This paper is concerned with the finite element method for nonlinear Hamiltonian systems from three aspects: conservation of energy, symplicity, and the global error. To study the symplecticity of the finite element methods, we use an analytical method rather than the commonly used algebraic method. We prove optimal order of convergence at the nodes t_n for mid-long time and demonstrate the symplecticity of high accuracy. The proofs depend strongly on superconvergence analysis. Numerical experiments show that the proposed method can preserve the energy very well and also can make the global trajectory error small for long time.

Mathematics subject classification: 65N30.
Key words: Nonlinear Hamiltonian systems, Finiteelement method, Superconvergence, Energy conservation, Symplecticity, Trajectory.

1. Introduction

We consider the nonlinear Hamiltonian systems

$$z_t = -JH_z, \quad z(0) = z_0, \tag{1.1}$$

where

$$H_z = \begin{pmatrix} H_p \\ H_q \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{1.2}$$

$z = (p, q)^T = (p_1, \cdots, p_n; q_1, \cdots, q_n)^T$, $H(z) = H(p, q)$ is a real-valued smooth function and J is a skew-symmetric matrix of order $2n$. Obviously, $J^T = J^{-1} = -J, \quad J^2 = -I_{2n}$. In application, the Hamiltonian $H(z)$ is often the total energy. Hamiltonian systems have two important properties: conservation and symplecticity. These properties are the hallmark of Hamiltonian systems.

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Symplectic geometry in phase space R^{2n} is the mathematical foundation to study Hamiltonian systems. Let $x = (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n})^T \in R^{2n}$. Then symplectic structure is defined by a skew-symmetric bilinear inner product

$$[x, y] = (x, Jy) = \sum_{j=1}^{n} (x_j y_{n+j} - x_{n+j} y_j), \qquad x, y \in R^{2n}. \tag{1.3}$$

Hence $[x, x] = (x, Jx) = 0$. In the symplectic space, a linear operator A is symplectic iff $A^T JA = J$. All solutions $z(t)$ of (1.1) form a symplectic group with one-parameter. These solutions have an important symplecticity property (see Section 5)

$$\Big(\frac{Dz(t)}{Dz_0}\Big)^T J \Big(\frac{Dz(t)}{Dz_0}\Big) = J, \qquad 0 \le t < \infty. \tag{1.4}$$

Moreover, multiplying Eq. (1.1) by J and z_t, we have the energy conservation

$$0 = \int_0^t J(z_t + JH_z) z_t dt = -\int_0^t H_z z_t dt = -H\big(z(t)\big)\Big|_0^t. \tag{1.5}$$

It is important to construct discrete algorithms which preserve these basic properties. Ruth [1] and Feng [2] have originally proposed the sympletic geometry algorithms which preserve the global symplectic structure and have tracking ability over long times. Feng and his co-authors then published several important works afterwards, see, e.g., [3-6]. Later on many symplectic schemes are studied by Chinese scholars, such as the partitioned algorithm (Sun [7]), multi-step algorithm (Tang [8]), volume-preserving algorithm (Shang [10,11]). Recent work can be found in [9,12,21] and a review [14].

Under the influence of Feng's work, several new symplectic algorithms are developed. For example, the symplectic Runge-Kutta method (SRK) is proposed by Sanz-Serna, Lasagni and Suris (see [15-18]). Later on the symplectic algorithms are also generalized to deal with partial differential systems.

Many scholars pointed out that the energy conservation is more important at certain times, see, e.g., Stuart *et al.* [19] (pp.583-584,642-644) and Hairer *et al.* [20] (p.12). So we turn to the finite element method (FEM). It is found that the continuous FEM always preserves the energy, and is approximately symplectic [22,23]. FEM is an exact orthogonal projection, which makes it possible to explore its refined properties, such as superconvergence, long-time error, approximate symplecticity and so on. These properties describe another kind of the structure different from the symplectic algorithms. Besides, the spectrum algorithm is also an orthogonal projection, see Tang and Xu [13]. It should be pointed out that the symplectic collocation method and SRK are equivalent under some conditions (see [20], p.27), which may be considered to be the approximately orthogonal projection based on some fixed quadrature. This quadrature makes the symplectic collocation method and SRK to possess the symplecticity, and to approximately preserve the energy. Therefore it is suggested that both SRK and FEM belong to the same setting of the orthogonal projection, but only with different quadratures.

In Table 1.1, we compare three properties of three algorithms: SFD (symplectic finite difference algorithm), SRK (symplectic Runge-Kutta method), and FEM (continuous finite element method).

In addition to preserving the symplecticity and energy, there is a third criterion to evaluate an algorithm, i.e., small deviations of computational trajectory after long times, which is possibly more important in applications. We now give a proposition as follows: