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ON HERMITIAN AND SKEW-HERMITIAN SPLITTING ITERATION METHODS FOR CONTINUOUS SYLVESTER EQUATIONS*

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Abstract

We present a *Hermitian and skew-Hermitian splitting* (**HSS**) iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite/semi-definite matrices. The unconditional convergence of the HSS iteration method is proved and an upper bound on the convergence rate is derived. Moreover, to reduce the computing cost, we establish an inexact variant of the HSS iteration method and analyze its convergence property in detail. Numerical results show that the HSS iteration method and its inexact variant are efficient and robust solvers for this class of continuous Sylvester equations.

Mathematics subject classification: 15A24, 15A30, 15A69, 65F10, 65F30, 65F50, 65H10
Key words: Continuous Sylvester equation, HSS iteration method, Inexact iteration, Convergence.

1. Introduction

We consider iterative solutions of the continuous Sylvester equations of the form

$$AX + XB = F, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{m\times n}$ are given complex matrices. Assume that

(A_1) A, B and F are large and sparse matrices;

(A_2) at least one of A and B is non-Hermitian; and

(A_3) both A and B are positive semi-definite, and at least one of them is positive definite.

Then from [14,29,31] we know that the continuous Sylvester equation (1.1) has a unique solution, as under the assumptions (A_1)-(A_3) there is no common eigenvalue between A and $-B$. Note that the continuous Lyapunov equation is a special case of the continuous Sylvester equation with $B = A^*$ and F Hermitian. Here and in the sequel, W^* is used to denote the conjugate transpose of the matrix $W \in \mathbb{C}^{m\times m}$, and we call W a positive definite or positive semi-definite matrix if so is its Hermitian part $\mathcal{H}(W) := \frac{1}{2}(W + W^*)$; note that a positive definite or positive semi-definite matrix is not necessarily Hermitian. We will also use $\mathcal{S}(W) := \frac{1}{2}(W - W^*)$ to denote the skew-Hermitian part of the matrix W. Obviously, it holds that $W = \mathcal{H}(W) + \mathcal{S}(W)$; see [2–6].

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The continuous Sylvester equation (1.1) has numerous applications in control and system theory [30,36,39], stability of linear systems [22], analysis of bilinear systems [32], power systems [25], linear algebra [16], signal processing [1], image restoration [11], filtering [21, 23], model order reduction [35], numerical methods for differential equations [8, 9], iterative methods for algebraic Riccati equations [7, 18–20, 30], matrix nearness problem [34, 41], finite element model updating [13,26], block-diagonalization of matrices [16,31] and so on. Many of these applications lead to stable Sylvester equations, i.e., Assumption (A_3) made in the above is satisfied.

The continuous Sylvester equation (1.1) is mathematically equivalent to the system of linear equations

$$\mathbf{A}x = f, \tag{1.2}$$

where $\mathbf{A} = I \otimes A + B^T \otimes I$, and the vectors x and f contain the concatenated columns of the matrices X and F, respectively, with $\otimes$ being the Kronecker product symbol and B^T representing the transpose of the matrix B. Of course, this is a numerically poor way to determine the solution X of the continuous Sylvester equation (1.1), as the system of linear equations (1.2) is costly to solve and can be ill-conditioned.

Standard methods for numerical solution of the continuous Sylvester equation (1.1) are the Bartels-Stewart and the Hessenberg-Schur methods [10, 15], which consist in transforming A and B into triangular or Hessenberg form by an orthogonal similarity transformation and then solving the resulting system of linear equations directly by a back-substitution process. These methods are classified as direct methods and are used, among others, by LAPACK and Matlab.

When the matrices A and B are large and sparse, iterative methods such as the Smith's method [37], the *alternating direction implicit* (**ADI**) method [11,24,33,40], the block *successive overrelaxation* (**BSOR**) method [38], the preconditioned conjugate gradient method [12], the matrix sign function method [27], and the matrix splitting methods [17] are often the methods of choice for efficiently and accurately solving the continuous Sylvester equation (1.1).

The Bartels-Stewart and the Hessenberg-Schur methods are applicable and effective for general continuous Sylvester equations of reasonably small sizes. For large and sparse continuous Sylvester equations, the afore-mentioned iterative methods are often superior to these direct methods, provided the matrices A and B are either Hermitian positive definite matrices or M-matrices. However, when the matrix A or B is not Hermitian, the convergence of these iterative methods may be theoretically not guaranteed, even if both matrices A and B are either asymptotically stable or N-stable (i.e., positive definite); this will be the case if the skew-Hermitian part of A or B is dominantly strong.

In this paper, we present an iterative method for solving the continuous Sylvester equation (1.1) by making use of the *Hermitian and skew-Hermitian* (**HS**) splittings of the matrices A and B. This *Hermitian and skew-Hermitian splitting* (**HSS**) iteration method is a matrix variant of the HSS iteration method firstly proposed in [6] for solving systems of linear equations, which are in spirit analogous to the ADI iteration methods [11, 24, 33, 40]. Via the HSS iteration method, the problem of solving a general continuous Sylvester equation is decomposed into a sequence of sub-problems about two coupled continuous Sylvester equations with respect to shifted Hermitian positive definite matrices and shifted skew-Hermitian matrices, respectively. When the matrices A and B are positive semi-definite, and at least one of them is positive definite, we prove that the HSS iteration converges unconditionally to the exact solution of the continuous Sylvester equation (1.1), with a bound on the convergence rate about the same as that of the conjugate gradient method when applied to a continuous Sylvester