Journal of Computational Mathematics
Vol.29, No.2, 2011, 215–226.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1009-m3246

A NOTE ON THE NONCONFORMING FINITE ELEMENTS FOR ELLIPTIC PROBLEMS*

Boran Gao
School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: ggggbr@163.com
Shuo Zhang
ICMSEC, AMSS, Chinese Academy of Science, Beijing 100190, PR China
Email: szhang@lsec.cc.ac.cn
Ming Wang
LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: mwang@math.pku.edu.cn

Abstract

In this paper, a class of rectangular finite elements for $2m$-th-oder elliptic boundary value problems in n-dimension ($m, n \geq 1$) is proposed in a canonical fashion, which includes the $(2m-1)$-th Hermite interpolation element ($n = 1$), the n-linear finite element ($m = 1$) and the Adini element ($m = 2$). A nonconforming triangular finite element for the plate bending problem, with convergent order $\mathcal{O}(h^2)$, is also proposed.

Mathematics subject classification: 65N30.
Key words: Nonconforming finite element, Elliptic boundary value problem, Plate bending problem.

1. Introduction

When the conforming finite element is used for numerically discretizing the elliptic problem, the convergence of the numerical solution to the exact solution depends on the approximation of the finite element space only. But the strong continuity requirement makes it difficult to construct such a conforming finite element. The idea of nonconforming finite element lies in that such difficulty can be overcome by loosing the request on the continuity. However, the loss of continuity will bring in the so-called consistent error, and some fundamental continuity of the finite element space is still necessary for well-posedness and convergence. This is the reason that most of the finite elements, conforming or nonconforming, were constructed case by case, depending on the order of the problem and sometimes the dimensions (cf. [1–3, 5, 7, 8, 12, 14, 15, 17]). A unified approach of constructing finite elements for general problems is still of theoretical and practical interest. Recently, a class of finite elements was discussed in a canonical fashion in [16], for all n-dimensional $2m$-th-order elliptic problem with $n \geq m \geq 1$. The well-known nonconforming linear element for the second-order problem and the Morley element for fourth-order problem are examples of this class. The class of finite elements is established on simplices, and makes use of the piecewise polynomials of the lowest degree. The nodal parameters are the natural ones to guarantee the fundamental continuity, and the consistency error can be controlled simultaneously.

* Received October 23, 2009 / Revised version received May 17, 2010 / Accepted July 20, 2010 / Published online November 20, 2010 /

In this paper, we will discuss the choice of nodal parameters that can be used to construct nonconforming finite elements, with admissible consistency error. We will first propose a class of rectangular finite elements for n-dimensional $2m$-th-oder problems $(m, n \geq 1)$ in a canonical fashion. The degrees of freedom are the values of function and all derivatives up to $(m-1)$-th-oder at all vertices of n-rectangle. The basic fundamental continuity is guaranteed and an $\mathcal{O}(h)$ convergence rate is shown. The $(2m-1)$-th Hermite interpolation element $(n=1)$, the n-linear finite element $(m=1)$ and the Adini element $(m=2)$ all belong to this class.

As almost all of the nonconforming finite elements are convergent in energy norm with order $\mathcal{O}(h)$, and the consistency error is the main limit, we will discuss the possibility of improving the convergence rate by strengthening the continuity of the finite element space. We choose the plate bending problem as an example. There have been successful attempts via other approaches, like conforming finite element, quasi-conforming finite elements (cf. [4,6,11]) and the double set parameter element (cf. [9]). But most nonconforming element for the plate bending problem, such as the Morley element [8], two Veubake elements [12], the NZT element [14], the rectangle Morley element (cf. [15]) and the Adini element (cf. [1]), are convergent with order $\mathcal{O}(h)$. In this work, a new nonconforming plate element will be given, with a convergence rate of $\mathcal{O}(h^2)$ in energy norm.

Finally, based on the new plate element, a new Zienkiewicz-type element will be deduced and reported for comparison. The new Zienkiewicz-type element is convergent for the plate bending problem with order $\mathcal{O}(h)$. Its consistent error is of order $\mathcal{O}(h^2)$ which is better than the two dimensional Zienkiewicz-type element proposed in [14]. In fact, the phenomenon that the consistency error can perform better than the approximation error has seldom been reported in literatures.

The paper is organized as follows. The rest of this section gives some basic notations. Section 2 gives the description of the class of rectangular finite elements. Section 3 gives the description of the new plate elements. Section 4 shows their convergence. Section 5 gives some numerical results for the new plate element.

Let n be a positive integer. Given a nonnegative integer k and a bounded domain $G \subset R^n$ with boundary ∂G, let $H^k(G)$, $H_0^k(G)$, $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ denote the usual Sobolev spaces, norm and semi-norm respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$.

We will use α, β, γ to denote n dimensional multi-indexes. Define

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

A finite element can be represented by a triple (T, P_T, D_T) with T the geometric shape, P_T the shape function space and D_T the vector of degrees of freedom, provided that D_T is P_T-unisolvent (see [5]).

Let Ω be a bounded polyhedron domain of R^n. For mesh size h with $h \to 0$, let $\mathcal{T}_h$ be a partition of Ω corresponding to a finite element (T, P_T, D_T), and let V_h, V_{h0} be the finite element spaces corresponding to the element and $\mathcal{T}_h$. Throughout this paper, we assume that $\{\mathcal{T}_h\}$ is shape regular.

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials defined on B with degree not greater than r, and $Q_r(B)$ the space of all polynomials with degree in each variable not greater than r.