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ERROR ESTIMATES OF THE FINITE ELEMENT METHOD WITH WEIGHTED BASIS FUNCTIONS FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION*

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Abstract

In this paper, we establish a convergence theory for a finite element method with weighted basis functions for solving singularly perturbed convection-diffusion equations. The stability of this finite element method is proved and an upper bound $\mathcal{O}(h|\ln\varepsilon|^{3/2})$ for errors in the approximate solutions in the energy norm is obtained on the triangular Bakhvalov-type mesh. Numerical results are presented to verify the stability and the convergent rate of this finite element method.

Mathematics subject classification: 65N30, 35J20.
Key words: Convergence, Singular perturbation, Convection-diffusion equation, Finite element method.

1. Introduction

It is known that singularly perturbed convection-diffusion problems contain sharp boundary layers so that the application of a standard finite element or finite difference method to such a problem often results in spurious oscillation. To avoid non-physical numerical solutions, many special finite element techniques have been developed, including upwind finite element [1, 4], Petrov-Galerkin finite element [7], streamline diffusion finite element methods [2, 8, 9], and exponentially fitted finite elements [18, 21–23]. However, these methods do not always give accurate results, especially when a diffusion coefficient has the same magnitude as that of mesh size. In [12], Li et al presented a weighted basis finite element method. Since the basis functions with weighted factors are consistent with the direction of flow and have the nature of exponential fitting near the boundary layers, numerical solutions obtained by applying this finite element method is non-oscillatory. Although the method proposed in [12] is promising from its numerical performance, except for a simple error bound of order $\mathcal{O}(h^{1/2}|\ln\varepsilon|)$ in [14] the mathematical understanding of the method is very limited. Regarding about the convergent results on layer-adapted meshes, streamline diffusion finite element or standard finite element methods can give uniformly optimal convergent rate, the reader is referred to [2,3,11,16,24–28]. Moreover, spectral methods have been proposed to resolve the bounding layers, which are shown very effective, see, e.g., [29, 30].

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In this work, a combination of the standard linear finite element method and the weighted basis finite element method is investigated for solving two-dimensional convection-dominated problems. This combination is used in conjunction with an anisotropic mesh refinement technique, i.e., a convection-diffusion equation is discretized by the weighted finite element method in a region containing the boundary layers and by the standard finite element method on a regular triangulation of the subregion away from the layers. As shown in [12], the standard basis function can be regarded as a special case of the weighted basis function. Therefore, this combination of two finite element methods is framed as a weighted basis finite element method which reduces to either the standard finite element or the weighted basis finite element method by a judicious choice of weights. Because the weighted basis functions are continuous across the interface between the two subregions, the resulting finite element space is conforming. This conformity allows us to analyze the method using conventional finite element analysis techniques. This is in contrast to a nonconforming method with which a sophisticated technique needs to be used to deal. In this paper, we will prove the stability of this finite element method and establish an upper error bound for the approximate solutions by the method on the triangular Bakhvalov-type mesh. We will also show that the error bound is almost independent of ε. We comment that, although the problem considered in this work is two-dimensional and linear, the idea can be extended to higher dimensional and/or nonlinear problems [5, 13].

Throughout this paper, we use C as a generic positive constant which is independent of the small parameter ε and the mesh size. The rest of our paper is organized as follows. Section 2 describes the continuous problems and some preliminaries. The finite element formulation with weighted basis functions is presented in Section 3. In Section 4, the stability of this finite element method is shown and the error estimate in an energy norm is established. The numerical examples will be given in Section 5 to demonstrate the convergent rate and the stability of this finite element method.

2. Weighted Basis Functions on the Triangular Mesh

Consider the following singularly perturbed problem with a small positive parameter ε in two-dimensional space,

$$\nabla \cdot (-\varepsilon \nabla v + \mathbf{b}(X)v) + \mu(X)v = f(X), \qquad X \in \Omega \subset \mathbb{R}^2, \tag{2.1}$$

$$v\big|_{\partial\Omega} = 0, \tag{2.2}$$

where $X = (x, y)^T$, $\Omega = (0,1) \times (0,1)$ and $\partial\Omega$ denotes the boundary of Ω.

In what follows, we will use conventional notation for function sets and spaces. More specifically, we use $L^2(\Omega)$ to denote the space of all square-integrable functions on Ω with the inner product $(\cdot,\cdot)$ and $C^k(\Omega)$ (or $C^k(\bar{\Omega})$) to denote the set of functions which, along with its up to kth derivatives are continuous on Ω (or $\bar{\Omega}$. The usual kth order Sobolev space is denoted by $H^k(\Omega)$ and we put $H_0^1(\Omega) = \{v \in H^1(\Omega) : v(X) = 0 \text{ on } \partial\Omega\}$.

For the coefficient functions, we assume that $\mathbf{b}(X) \in (C^1(\bar{\Omega}))^2$, $\mu(X) \in C(\bar{\Omega}) \cap H^1(\Omega)$ and $f(X) \in L^\infty(\Omega)$. We also assume that $\mathbf{b}(X)$ satisfies

$$\frac{1}{2}\nabla \cdot \mathbf{b} + \mu(X) \geq \alpha > 0, \qquad X \in \Omega, \tag{2.3}$$

where α is a positive constant. This condition (2.3) has been used in many existing works on uniform convergence analysis such as [18–20, 22, 25]. In fact, when ε is sufficiently small, the