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HIGH ORDER NUMERICAL METHODS TO TWO DIMENSIONAL HEAVISIDE FUNCTION INTEGRALS*

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Abstract

In this paper we design and analyze a class of high order numerical methods to two dimensional Heaviside function integrals. Inspired by our high order numerical methods to two dimensional delta function integrals [19], the methods comprise approximating the mesh cell restrictions of the Heaviside function integral. In each mesh cell the two dimensional Heaviside function integral can be rewritten as a one dimensional ordinary integral with the integrand being a one dimensional Heaviside function integral which is smooth on several subsets of the integral interval. Thus the two dimensional Heaviside function integral is approximated by applying standard one dimensional high order numerical quadratures and high order numerical methods to one dimensional Heaviside function integrals. We establish error estimates for the method which show that the method can achieve any desired accuracy by assigning the corresponding accuracy to the sub-algorithms. Numerical examples are presented showing that the second to fourth-order methods implemented in this paper achieve or exceed the expected accuracy.

Mathematics subject classification: 65D05, 65D30, 65D32.
Key words: Heaviside function integral, High order numerical method, Irregular domain.

1. Introduction

We study in this paper a class of high order numerical methods to the two dimensional Heaviside function integrals

$$\int_{\mathbb{R}^2} f(x,y)H(u(x,y))dxdy, \tag{1.1}$$

where $f(x,y)$ is an integrand function, $u(x,y)$ is a level set function whose zero points define certain curve in the two dimensional space which compose the boundaries of an irregular bounded domain $\Omega = \{(x,y)|u(x,y) > 0\}$. The Heaviside function integral (1.1) is equivalent to

$$\int_{\Omega} f(x,y)dxdy. \tag{1.2}$$

We consider that the functions f, u have sufficient smoothness and their values are only provided at grid points of a regular mesh. The domain Ω is defined implicitly by the level set function u. Studying the computations of Heaviside function integrals in two and three dimensions in the above context is applicable to many problems. One example is computing immiscible multiphase flow [2, 9, 11]. In such applications the unknown quantities such as density and viscosity are generally discontinuous across interfaces separating the immiscible fluids. One convenient strategy is to employ fixed computational mesh and allow the moving interface to cut through mesh cells. In this situation the computations by finite element method requires

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evaluating integrals with discontinuous integrands in the variational formulation, which can be performed by resorting to computations of Heaviside function integrals.

The computation of Heaviside function integrals has much relation with computing delta function integrals. The latter problem corresponds to evaluating integrals restricted on the domain boundary $\partial\Omega$ which is a codimension one manifold. For the study of numerical methods to delta function integrals one can refer to [1, 3–5, 8, 10, 12–14, 16–20]. Most of these methods have also been extended to study the computations of Heaviside function integrals.

In [10] Tornberg studied the computations of two dimensional Heaviside and delta function integrals. The approach is to regularize the discontinuous or singular integrand, and then apply a standard quadrature to the integral with the regularized integrand. This approach allows the error analysis by separately considering the analytical error from regularization and numerical error from quadrature. The error of the approach is determined by the moment and regularity conditions of the regularized delta function and the order of the quadrature method. This approach can be designed to be of arbitrary high order accuracy. However high order method requires utilizing regularized delta functions with very high order moment or regularity conditions which can be complicated and may influence the efficiency of the method.

In [3] Engquist, Tornberg and Tsai studied the regularization of multidimensional Heaviside function based on regularized one dimensional Heaviside functions and a variable support size formula. The method is shown to be second-order accurate which improves on the first-order accuracy of the conventional regularization based on regularized one dimensional Heaviside functions and the constant support size formula. They also presented a regularized Heaviside function based on integrating a product formula of one dimensional discrete delta functions. The product formula method following Peskin [6, 7] has the advantage that it can achieve any desired accuracy by using one dimensional discrete delta functions with corresponding discrete moment conditions (see the proof in [12]). However the high order version of the product formula method has not been implemented in the case of the domain Ω implicitly defined by a level set function.

In [4, 5] Min and Gibou designed a geometric integration method for computing Heaviside and delta function integrals. The approach to Heaviside function integrals is to decompose the domain Ω into simplices on which the numerical quadrature can be applied. This method gives second-order results.

In [15] Towers proposed a type of methods for discretizing multidimensional Heaviside function based on approximating the Heaviside function by finite differencing its first few primitives. This idea has been adopted to study the approximation of delta function integrals in [13, 14, 16]. Two variants of the methods are presented in [15] for computing Heaviside function integrals. The first method gives second-order accuracy. The second method is shown to give third-order accuracy for a specific one dimensional example and behave fourth-order convergent for general multidimensional computations. Error analysis for the second-order version method is given in [15]. We will give a comparison between the numerical results of the high order version method in [15] and our high order methods in this paper, which shows the advantage of our third- and fourth-order methods.

In this paper we design and analyze a class of high order numerical methods to the two dimensional Heaviside function integrals (1.1). We construct these methods by considering the approximation of the restriction of the two dimensional Heaviside function integral in each mesh cell. Such a natural strategy of approximating mesh cell restrictions of integrals has already been adopted in [17] for designing high order methods to delta function integrals of full codimension