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ON EIGENVALUE BOUNDS AND ITERATION METHODS FOR DISCRETE ALGEBRAIC RICCATI EQUATIONS*

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Abstract

We derive new and tight bounds about the eigenvalues and certain sums of the eigenvalues for the unique symmetric positive definite solutions of the discrete algebraic Riccati equations. These bounds considerably improve the existing ones and treat the cases that have been not discussed in the literature. Besides, they also result in completions for the available bounds about the extremal eigenvalues and the traces of the solutions of the discrete algebraic Riccati equations. We study the fixed-point iteration methods for computing the symmetric positive definite solutions of the discrete algebraic Riccati equations and establish their general convergence theory. By making use of the Schulz iteration to partially avoid computing the matrix inversions, we present effective variants of the fixed-point iterations, prove their monotone convergence and estimate their asymptotic convergence rates. Numerical results show that the modified fixed-point iteration methods are feasible and effective solvers for computing the symmetric positive definite solutions of the discrete algebraic Riccati equations.

Mathematics subject classification: 15A15, 15A18, 15A24, 15A48, 65F30.
Key words: Discrete algebraic Riccati equation, Symmetric positive definite solution, Eigenvalue bound, Fixed-point iteration, Convergence theory.

1. Introduction

Consider the *discrete algebraic Riccati equation* (**DARE**)

$$X = A^T XA - A^T XB(G + B^T XB)^{-1}B^T XA + C^T C, \tag{1.1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$ and $G \in \mathbf{R}^{m\times m}$ are given matrices, and the matrix G is assumed to be symmetric and positive definite. Let

$$R = BG^{-1}B^T \quad \text{and} \quad Q = C^T C. \tag{1.2}$$

Then by applying the Sherman-Morrison-Woodbury formula [10, P. 50], the DARE (1.1) can be equivalently reformulated as

$$X = A^T X(I + RX)^{-1}A + Q,$$

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where I represents the identity matrix of appropriate size, and R and Q are the matrices defined in (1.2) satisfying $R \succeq 0$ and $Q \succeq 0$. Here and in the sequel, for a square matrix W we say $W \succ 0$ (or $W \succeq 0$) if W is symmetric positive definite (or symmetric positive semidefinite).

Throughout the paper we assume that (A, B) is a stabilizable pair and (A, C) is a detectable pair [1)] . Then the DARE (1.1) has a unique symmetric positive definite solution X such that the matrix $(I + RX)^{-1}A$ is stable, i.e., every eigenvalue λ of the matrix $(I + RX)^{-1}A$ satisfies $|\lambda| < 1$; see, e.g., [24,44]. Under this assumption, the DARE (1.1) can be further rewritten in the symmetric form as

$$X = A^T(X^{-1} + R)^{-1}A + Q. \tag{1.3}$$

In this paper, we will focus on discussions about the discrete algebraic Riccati equations of the form (1.3).

The discrete algebraic Riccati equation (1.1) arises in many areas of engineering applications such as the optimal control design [21] and the filter design [2]. One typical and important application about the DARE (1.1) is the discrete-time LQ-problem in optimal control. Under the assumption that the matrix pairs (A, B) is stabilizable and (A, C) is detectable, the discrete-time linear system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, & x_0 \text{ given}, \\ y_k = Cx_k, & k = 0, 1, 2, \cdots, \end{cases}$$

exists an optimal control u_k, which is the minimizer of the quadratic cost functional

$$J = \sum_{k=0}^{\infty} \Big(x_k^T Q x_k + u_k^T G u_k \Big).$$

Then u_k can be recovered via x_k by

$$u_k = -(G + B^T XB)^{-1}B^T XAx_k, \qquad k = 0, 1, 2, \cdots,$$

where X is the unique symmetric positive definite solution of the DARE (1.1). We remark that when the above-mentioned linear system is subjected to perturbations, uncertainties, additive/multiplicative noises or a time delay, the DARE (1.3) may be appropriately modified and is often impossible to be solved exactly.

An accurate estimate about the solution of the DARE (1.3) or, equivalently, the DARE (1.1), is theoretically important and practically useful when we treat some control problems such as the stabilized control design for time-delay systems [29], the stability analysis in the presentations of time delay and perturbations [43], and the state and error covariance estimation [20], as well as when we select feasible starting points for certain iteration methods employed to solve the discrete algebraic Riccati equations.

In fact, a bound for the solution X of the DARE (1.3) can be provided through a bound on the eigenvalues $\lambda_i(X)$ of X. Various bounds about the extreme eigenvalues [9], the partial sum and the partial product of eigenvalues [17,19], the trace [13,22,38], and the determinant [42] of the solution X have been derived during the past three decades; see [23,37] for excellent

[1)] For a complex constant λ and vector w, if $w^*B = 0$ and $w^*A = \lambda w^*$ imply either $|\lambda| < 1$ or $w = 0$, then the matrix pair (A, B) is called stabilizable. The matrix pair (A, C) is called detectable if (A^T, C^T) is stabilizable. Here, $(\cdot)^T$ and $(\cdot)^*$ denote the transpose and the conjugate transpose of either a complex vector or a complex matrix, respectively.