Journal of Computational Mathematics
Vol.29, No.4, 2011, 415–440.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1101-m2996

# A NEW STABILIZED SUBGRID EDDY VISCOSITY METHOD BASED ON PRESSURE PROJECTION AND EXTRAPOLATED TRAPEZOIDAL RULE FOR THE TRANSIENT NAVIER-STOKES EQUATIONS*

Minfu Feng
*School of Mathematics, Sichuan University, Chengdu 610064, China*
*Email: fmf@wtjs.cn*
Yanhong Bai
*Department of Mathematics, Taiyuan Teachers College, Taiyuan 030012, China*
*Email: baiyanhong1982@126.com*
Yinnian He
*Faculty of Science, Xian Jiaotong University, Xian 710049, China*
Yanmei Qin
*Key Laboratory of Numberical Simulation of Sichuan Province, Neijiang 641112, China*

**Abstract**

We consider a new subgrid eddy viscosity method based on pressure projection and extrapolated trapezoidal rule for the transient Navier-Stokes equations by using lowest equal-order pair of finite elements. The scheme stabilizes convection dominated problems and ameliorates the restrictive inf-sup compatibility stability. It has some attractive features including parameter free for the pressure stabilized term and calculations required for higher order derivatives. Moreover, it requires only the solutions of the linear system arising from an Oseen problem per time step and has second order temporal accuracy. The method achieves optimal accuracy with respect to solution regularity.

*Mathematics subject classification:* 65N30, 76D05.
*Key words:* Subgrid eddy viscosity model, Pressure projection method, Extrapolated trapezoidal rule, The transient Navier-Stokes equations.

## 1. Introduction

The flow of an incompressible fluid is governed by the incompressible Navier-Stokes equations

$$\begin{aligned}
&\mathbf{u}_t - \nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, && \text{in} \quad (0,T]\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0, && \text{in} \quad [0,T]\times\Omega,\\
&\mathbf{u} = 0, && \text{in} \quad (0,T]\times\partial\Omega,\\
&\mathbf{u}(0,x) = \mathbf{u}_0, && \text{in} \quad \Omega,\\
&\int_\Omega p\ dx = 0, && \text{in} \quad (0,T],
\end{aligned} \tag{1.1}$$

where $\Omega\subset R^2$ is a bounded domain with boundary $\partial\Omega$, $[0,T]$ is a finite time interval, $\mathbf{u}(t,x)$ is the velocity of the fluid and $p(t,x)$ is the pressure. The viscosity $\nu>0$, which is inverse proportional to the Reynolds number $Re=\mathcal{O}(\nu^{-1})$. The body forces $\mathbf{f}(t,x)$ and the initial velocity

* Received December 13, 2008 / Revised version received October 26, 2010 / Accepted December 22, 2010 / Published online June 27, 2011 /

field $\mathbf{u}_0$ are given. Generally speaking, for the transient Navier-Stokes equations which govern viscous fluid flow, the natural and important Galerkin approximation is a mixed method, however the Galerkin mixed finite element approximation of (1.1) may suffer from three problems: violation of the discrete inf-sup (or Babuska-Brezzi) stability condition, dominating advection, and how to make fully discretization which is a simple, second order temporal accuracy.

The subgrid eddy viscosity model is a numerical stabilization of a convection dominated and underresolved flow. This approach adds an artificial viscosity only on the fine scales, and is referred to artificial viscosity model, which is inspired by earlier work of Guermend [4]. In [4] subgrid scales are augmented by bubble functions. Later, Layton generalized the concept for the stationary convection diffusion problem. In the work of Kaya and Layton [9], this model has been connected with another consistent stabilization technique, also known as variational multiscale method. The model has been analyzed for time-dependent Navier-Stokes equations by John-Kaya [8] and Kaya-*Rivière* [10]. In [11], Kaya-*Rivière* gave algorithm and numerical experiments for variational multiscale method. However, these works require velocity and pressure finite element spaces satisfying the so-called inf-sup condition.

It is well known that the simplest conforming low-order elements like $P_1 - P_1$ triangular element is not stable. This impacts on efficiency, since local mass conservation, the simple logic and regular data structure associate with low-order finite element methods are very attractive and useful on many occasions. To counteract the lack of LBB stability, low-order pairs are usually supplemented by stabilized procedures. Stabilized mixed finite element methods are often developed by using residuals of the momentum equation, e.g., Douglas-Wang method [2], least squares Petrov-Galerkin finite element method [14]. These residual terms must be formulated using mesh-dependent parameters, whose optimal values are usually unknown. Particularly, pressure and velocity derivatives in this residual vanish or are poorly approximated, causing difficulties in the application of consistent stabilization. Other stabilized mixed methods involving non-residual stabilization are also developed, e.g., pressure projection method, it has been applied to the Stokes problem by Bochev [1]; He-Li [6], Li-He-Chen [12] extended this method to the Navier-Stokes problem. Pressure projection method does not require approximation of derivatives, specification of mesh-dependent parameter, or nonstandard data structures. The paper [12] only counteracted the lack of LBB stability condition and made a semi-discrete analysis; the solution has oscillation when the viscosity coefficient is small.

When (1.1) is fully-discretized by accurate and stable methods, well stabilized methods with second-order temporal accuracy are Crank-Nicolson scheme (see Heywood and Rannacher [7]), Crank-Nicolson extrapolation scheme (see Girault and Raviart [3]), and two-level method based on finite element and Crank-Nicolson extrapolation (see He [5]). However, all these discrete forms are nonlinear, and the approximation can still fail for many reasons. One common mode of failure is non-convergence of the iterative nonlinear and linear solvers used to compute the velocity and pressure at the new time levels. We consider herein a simple, second order accurate, and stable method for temporal discretization which addresses the failure cases mentioned above. The method requires the solution of one linear system per time step.

In this paper, we propose a new stabilization finite element method which is combined subgrid eddy viscosity with pressure projection method for the spatial discretization and extrapolated trapezoidal rule for the temporal discretization by using lowest equal-order pair of finite elements. The scheme stabilizes convection domination and ameliorates the restrictive inf-sup compatibility stability, which has second order temporal accuracy of $\mathcal{O}(\triangle t^2 + \nu_T^{\frac{1}{2}} H + h)$, where the constant in the estimate does not depend on the Reynolds number but on the reduced