HSS METHOD WITH A COMPLEX PARAMETER FOR THE SOLUTION OF COMPLEX LINEAR SYSTEM

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Abstract
In this paper, a complex parameter is employed in the Hermitian and skew-Hermitian splitting (HSS) method (Bai, Golub and Ng: SIAM J. Matrix Anal. Appl., 24(2003), 603-626) for solving the complex linear system $Ax = f$. The convergence of the resulting method is proved when the spectrum of the matrix $A$ lies in the right upper (or lower) part of the complex plane. We also derive an upper bound of the spectral radius of the HSS iteration matrix, and an estimated optimal parameter $\alpha$ (denoted by $\alpha_{est}$) of this upper bound is presented. Numerical experiments on two modified model problems show that the HSS method with $\alpha_{est}$ has a smaller spectral radius than that with the real parameter which minimizes the corresponding upper bound. In particular, for the 'dominant' imaginary part of the matrix $A$, this improvement is considerable. We also test the GMRES method preconditioned by the HSS preconditioning matrix with our parameter $\alpha_{est}$.

Key words: Hermitian matrix, Skew-Hermitian matrix, Splitting iteration method, Complex linear system, Complex parameter.

1. Introduction
We are interested in the iterative solution of the following complex linear system
\[ Ax = f. \tag{1.1} \]
We consider the case in which $A \in C^{n \times n}$ is large, sparse, non-Hermitian and positive definite and $f \in C^n$; see several applications in [12,17,21].

Bai, Golub and Ng [6] proposed the Hermitian/skew-Hermitian splitting (HSS) method based on the fact that the matrix $A$ naturally possesses the Hermitian/skew-Hermitian splitting
\[ A = H + S, \]
where $H = \frac{1}{2}(A + A^H)$ is the Hermitian matrix, $S = \frac{1}{2}(A - A^H)$ is the skew-Hermitian matrix, and $A^H$ is the conjugate transpose of the matrix $A$. The HSS method has the following form:
\[
\begin{align*}
(\alpha I + H)x^{(k+1)} &= (\alpha I - S)x^{(k)} + f, \\
(\alpha I + S)x^{(k+1)} &= (\alpha I - H)x^{(k+1)} + f,
\end{align*}
\tag{1.2}
\]
where the parameter $\alpha > 0$ can be chosen. The above form can be equivalently rewritten as
\[
x^{(k+1)} = T(\alpha)x^{(k)} + G(\alpha)f, \quad k = 0, 1, 2, \ldots, \tag{1.3}
\]
where $T(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$ is the iteration matrix, and $G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}$.

The following theorem [6] gives the convergence property of the HSS iteration.

**Theorem 1.1.** Suppose that $A \in \mathbb{C}^{n \times n}$ is a positive definite matrix, $H = \frac{1}{2}(A + A^H), S = \frac{1}{2}(A - A^H)$ are the Hermitian and Skew-Hermitian parts of $A$ respectively, and the parameter $\alpha > 0$. Then the spectral radius $\rho(T(\alpha))$ of the iteration matrix $T(\alpha)$ of the HSS iteration is bounded by

$$\rho(T(\alpha)) \leq \sigma(\alpha) = \max_{\lambda_j \in \Lambda(H)} \frac{\alpha - \lambda_j}{\alpha + \lambda_j},$$

(1.4)

where $\Lambda(\cdot)$ represents the spectrum of the corresponding matrix. Since $A$ is positive definite ($\lambda_j > 0$), we have

$$\rho(T(\alpha)) \leq \sigma(\alpha) < 1, \quad \text{for all} \quad \alpha > 0,$$

i.e., the HSS iteration converges.

Furthermore, let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of $H$. Then the upper bound $\sigma(\alpha)$ has the optimal parameter

$$\tilde{\alpha} = \sqrt{\lambda_1\lambda_n}$$

(1.5)

and

$$\sigma(\tilde{\alpha}) = \min_{\alpha > 0} \sigma(\alpha) = \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1},$$

where $\kappa(H) = \frac{\lambda_1}{\lambda_n}$ is the spectral condition number of $H$.

However, we have the following observations:

1. $\tilde{\alpha}$ is usually different from the optimal parameter $\alpha^* = \arg \min_{\alpha > 0} \rho(T(\alpha))$.
2. Numerical experiments in [5,6,10,11] have shown that in most situations,

$$\rho(T(\alpha^*)) \ll \rho(T(\tilde{\alpha})).$$

3. $\tilde{\alpha}$ and $\sigma(\tilde{\alpha})$ do not include any information of $S$.

To further improve the efficiency of the HSS method, it is desirable to determine or find a good estimate for the optimal parameter $\alpha^*$. For some special constructed matrices, in particular, for saddle-point problems, the optimal parameter, or the quasi-optimal parameter [2], has been extensively discussed [2,4,5,8,10], and the results show that the optimal parameter does include the information of $S$.

The matrix

$$P = \frac{1}{2\alpha}(H + \alpha I)(S + \alpha I)$$

(1.6)

can also be employed as a preconditioner [2,8,11,23], where $\alpha$ is referred to as the preconditioning parameter. The idea of HSS preconditioner is motivated from the HSS method.

More generally, the coefficient matrix $A \in \mathbb{C}^{n \times n}$ can be splitted into

$$A = N + S_0,$$

where $N$ is a normal matrix and $S_0$ is a skew-Hermitian matrix. Similarly to the HSS method, normal/skew-Hermitian splitting (NSS) method with a real parameter could be formed [7].