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HSS METHOD WITH A COMPLEX PARAMETER FOR THE SOLUTION OF COMPLEX LINEAR SYSTEM*

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Abstract

In this paper, a complex parameter is employed in the Hermitian and skew-Hermitian splitting (HSS) method (Bai, Golub and Ng: SIAM J. Matrix Anal. Appl., 24(2003), 603-626) for solving the complex linear system $Ax = f$. The convergence of the resulting method is proved when the spectrum of the matrix A lie in the right upper (or lower) part of the complex plane. We also derive an upper bound of the spectral radius of the HSS iteration matrix, and a estimated optimal parameter α (denoted by α_{est}) of this upper bound is presented. Numerical experiments on two modified model problems show that the HSS method with α_{est} has a smaller spectral radius than that with the real parameter which minimizes the corresponding upper bound. In particular, for the $'$dominant$'$ imaginary part of the matrix A, this improvement is considerable. We also test the GMRES method preconditioned by the HSS preconditioning matrix with our parameter α_{est}.

Mathematics subject classification: 65F10,65Y20.
Key words: Hermitian matrix, Skew-Hermitian matrix, Splitting iteration method, Complex linear system, Complex parameter.

1. Introduction

We are interested in the iterative solution of the following complex linear system

$$Ax = f. \tag{1.1}$$

We consider the case in which $A \in C^{n\times n}$ is large, sparse, non-Hermitian and positive definite and $f \in C^n$; see several applications in [12,17,21].

Bai, Golub and Ng [6] proposed the Hermitian/skew-Hermitian splitting (HSS) method based on the fact that the matrix A naturally possesses the Hermitian/skew-Hermitian splitting

$$A = H + S,$$

where $H = \frac{1}{2}(A + A^H)$ is the Hermitian matrix, $S = \frac{1}{2}(A - A^H)$ is the skew-Hermitian matrix, and A^H is the conjugate transpose of the matrix A. The HSS method has the following form:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + f, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + f, \end{cases} \tag{1.2}$$

where the parameter $\alpha > 0$ can be chosen. The above form can be equivalently rewritten as

$$x^{(k+1)} = T(\alpha)x^{(k)} + G(\alpha)f, \qquad k = 0, 1, 2, \cdots, \tag{1.3}$$

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where $T(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$ is the iteration matrix, and $G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}$.

The following theorem [6] gives the convergence property of the HSS iteration.

Theorem 1.1. *Suppose that $A \in C^{n\times n}$ is a positive definite matrix, $H = \frac{1}{2}(A + A^H), S = \frac{1}{2}(A - A^H)$ are the Hermitian and Skew-Hermitian parts of A respectively, and the parameter $\alpha > 0$. Then the spectral radius $\rho(T(\alpha))$ of the iteration matrix $T(\alpha)$ of the HSS iteration is bounded by*

$$\rho(T(\alpha)) \le \sigma(\alpha) = \max_{\lambda_j \in \Lambda(H)} \left| \frac{\alpha - \lambda_j}{\alpha + \lambda_j} \right|, \tag{1.4}$$

where $\Lambda(\cdot)$ represents the spectrum of the corresponding matrix. Since A is positive definite ($\lambda_j > 0$), we have

$$\rho(T(\alpha)) \le \sigma(\alpha) < 1, \qquad \text{for all} \;\; \alpha > 0,$$

i.e., the HSS iteration converges.

Furthermore, let $\lambda_1 \ge \cdots \ge \lambda_n > 0$ be the eigenvalues of H. Then the upper bound $\sigma(\alpha)$ has the optimal parameter

$$\tilde{\alpha} = \sqrt{\lambda_1 \lambda_n} \tag{1.5}$$

and

$$\sigma(\tilde{\alpha}) = \min_{\alpha>0} \sigma(\alpha) = \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1},$$

where $\kappa(H) = \frac{\lambda_1}{\lambda_n}$ is the spectral condition number of H.

However, we have the following observations:

(1) $\tilde{\alpha}$ is usually different from the optimal parameter

$$\alpha^* = arg \min_{\alpha>0} \rho(T(\alpha)).$$

(2) Numerical experiments in [5,6,10,11] have shown that in most situations,

$$\rho(T(\alpha^*)) \ll \rho(T(\tilde{\alpha})).$$

(3) $\tilde{\alpha}$ and $\sigma(\tilde{\alpha})$ do not include any information of S.

To further improve the efficiency of the HSS method, it is desirable to determine or find a good estimate for the optimal parameter α^*. For some special constructed matrices, in particular, for saddle-point problems, the optimal parameter, or the quasi-optimal parameter [2], has been extensively discussed [2,4,5,8,10], and the results show that the optimal parameter does include the information of S.

The matrix

$$P = \frac{1}{2\alpha}(H + \alpha I)(S + \alpha I) \tag{1.6}$$

can also be employed as a preconditioner [2,8,11,23], where α is referred to as the preconditioning parameter. The idea of HSS preconditioner is motivated from the HSS method.

More generally, the coefficient matrix $A \in C^{n\times n}$ can be splitted into

$$A = N + S_0,$$

where N is a normal matrix and S_0 is a skew-Hermitian matrix. Similarly to the HSS method, *normal/skew-Hermitian splitting* (NSS) method with a real parameter could be formed [7].