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RATIONAL SPECTRAL COLLOCATION METHOD FOR A COUPLED SYSTEM OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS*

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Abstract

A novel collocation method for a coupled system of singularly perturbed linear equations is presented. This method is based on rational spectral collocation method in barycentric form with sinh transform. By sinh transform, the original Chebyshev points are mapped into the transformed ones clustered near the singular points of the solution. The results from asymptotic analysis about the singularity solution are employed to determine the parameters in this sinh transform. Numerical experiments are carried out to demonstrate the high accuracy and efficiency of our method.

Mathematics subject classification: 65L10, 65M70.
Key words: Singular perturbation, Coupled system, Rational spectral collocation method, Boundary layer, Reaction-diffusion, Convection-diffusion.

1. Introduction

In this paper, we consider a coupled system of $m \geq 2$ singularly perturbed linear equations in the unknown vector function $\mathbf{u} = (u_1, \cdots, u_m)^T$. This system is coupled through its convective and reactive terms:

$$\mathcal{L}\ \mathbf{u} := -\varepsilon \mathbf{u}'' - B(x)\mathbf{u}' + A(x)\mathbf{u} = \mathbf{f}, \tag{1.1}$$

and it satisfies the boundary conditions

$$\mathbf{u}(0) = \mathbf{b_0}, \quad \mathbf{u(1)} = \mathbf{b_1}. \tag{1.2}$$

Here $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times m$ matrices whose entries are assumed to lie in $C^2[0,1]$, and $\varepsilon > 0$ is a small diffusion parameter whose presence makes the problem singularly perturbed. We assumed that $\mathbf{f} = (f_1, \cdots, f_m)^T \in \left(C^2[0,1]\right)^m$, both $\mathbf{b_0} = (b_{01}, \cdots, b_{0m})^T$ and $\mathbf{b_1} = (b_{11}, \cdots, b_{1m})^T$ are constant vectors.

Coupled systems do appear in many applications , notably turbulent interaction of waves and currents [1], diffusion processes in electroanalytic chemistry [2], optimal control and certain resistance-capacitor electrical circuits [3], etc. Compared to single-equation singularly perturbed problems, coupled systems can model more complicated physical phenomena.

If $B \equiv 0$ in (1.1), the system is said to be of reaction-diffusion type. Shishkin [2] established finite difference method on a piecewise uniform mesh for the case $m = 2$; the results about the stability, convergence and error estimate for Shishkin's method, can be found in Madden [4], Linβ [5, 6], Matthews [7]. Stephens [8] proposed a parameter-uniform overlapping Schwarz

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method and Linβ [9] established a central difference scheme on certain layer-adapted meshes for the cases of $m \geq 2$. Generally speaking, the solution to the problem of this type has two boundary layers with width $\mathcal{O}(\sqrt{\varepsilon})$ respectively at $x = 0$ and $x = 1$ under some assumptions.

If $B \not\equiv 0$, the system is said to be of convection-diffusion type. Riordan et al presented a finite difference method consisting of upwinding on piecewise-uniform Shishkin meshes for the cases of $m = 2$ [10] and $m > 2$ [11]. They also used a Jacobi-type iteration to compute the solution [12]. Generally speaking, the problem of this type has a solution with a single boundary layer of width $\mathcal{O}(\varepsilon)$ at $x = 0$ (or $x = 1$) under proper assumptions.

Furthermore, if (1.1) are coupled through their convective (first-order) terms (i.e. for each $i = 1, \cdots, m$ there exits a $j \neq i$ such that $b_{ij} \neq 0$), we say it is strongly coupled; otherwise, if $B \equiv 0$ or B is just a non-zero diagonal matrix, it is said to be weakly coupled.

The spectral collocation method based on rational interpolants in barycentric form was proposed by Berrut and his collaborators [13–16]. An advantage of which is that after transform, the derivatives in underlying differential equation are not required to be transformed correspondingly as is usual in other methods. Besides, Tee and Trefethen [17] devised a sinh transform that maps original Chebyshev points clustered near the boundaries of $[-1, 1]$ into a new set of collocation points, say the transformed Chebyshev points, which are clustered near the singular point of a function. In order to determine the parameters in above sinh transform, singularities of the solution, including the location and width of the boundary layer, should be known. Hence, we resorted to the singularity location technique in asymptotic analysis and solved a parameterized singular perturbation problem [18].

Here we present a kind of numerical method based on rational spectral collocation in barycentric form with sinh transform (RSC-sinh method) for solving a coupled system of singularly perturbed problems in various types, both weakly coupled and strongly coupled. Numerical experiments illustrate that the RSC-sinh method enjoys improved spectral accuracy.

This paper is organized as follows. The asymptotic analysis of the problem is outlined in Section 2. In Section 3, we construct the RSC-sinh method for problem (1.1)-(1.2). The numerical results of several examples are given in Section 4. Finally, we present some concluding remarks in Section 5.

Notation. Through out this paper, C denotes a generic constant that may take different values at different places in our arguments. The definition of norm is:

- for a vector $\mathbf{y} = (y_0, \cdots, y_m)^T$, $\|\mathbf{y}\| = \max_{p=0,\cdots,m} |y_p|$;
- for a real-valued function $y \in C([0,1])$, $\|y\|_{[0,1]} = \max_{x\in[0,1]} |y(x)|$;
- for a vector-valued function $\mathbf{z} = (z_0, \cdots, z_m)^T$, $\|\mathbf{z}\|_{[0,1]} = \max_{p=0,\cdots,m} \|z_p\|_{[0,1]}$.

2. Asymptotic Analysis

For the construction of RSC-sinh method, it is necessary to have properties of exact solution, especially the location and width of the boundary layer. The asymptotic analysis of problem (1.1) often involves a *Shishkin Decomposition* [19], which splits the solution into regular and layer components. It will be considered in reaction-diffusion case and convection-diffusion case respectively.