Journal of Computational Mathematics
Vol.29, No.5, 2011, 491–500.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1105-m3392

# A HIGH ORDER ADAPTIVE FINITE ELEMENT METHOD FOR SOLVING NONLINEAR HYPERBOLIC CONSERVATION LAWS*

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**Abstract**

In this note, we apply the $h$-adaptive streamline diffusion finite element method with a small mesh-dependent artificial viscosity to solve nonlinear hyperbolic partial differential equations, with the objective of achieving high order accuracy and mesh efficiency. We compute the numerical solution to a steady state Burgers equation and the solution to a converging-diverging nozzle problem. The computational results verify that, by suitably choosing the artificial viscosity coefficient and applying the adaptive strategy based on *a posterior* error estimate by Johnson *et al.*, an order of $N^{-3/2}$ accuracy can be obtained when continuous piecewise linear elements are used, where $N$ is the number of elements.

*Mathematics subject classification:* 65N30.
*Key words:* Adaptive finite element, Nonlinear hyperbolic conservation law.

## 1. Introduction

For the nonlinear hyperbolic equations

$$u_t + f(u)_x = 0 \tag{1.1}$$

discontinuities may develop after finite time despite the regularity of the initial condition. This accounts for most of the difficulties in the design of numerical schemes to solve the nonlinear hyperbolic equation (1.1) accurately. It has been shown [6, 7] that the numerical solution generated by high order methods produces in general only first order accuracy for pointwise errors, because the information carried along characteristics is degraded to first order when passing through the discontinuity. Much work has been done in the literature to achieve high order accuracy in the presence of discontinuities. To name a few, pre- and post-processing has been introduced in [6,7] to recover high order pointwise accuracy for linear hyperbolic systems; Glimm and his co-authors [2] have designed high order front tracking algorithms; Gottlieb et al. [3] applied the Gegenbauer postprocessing to recover the designed accuracy up to the shock front in the framework of high order weighted essentially non-oscillatory (WENO) schemes. In

* Received April 21, 2010 / Revised version received September 24, 2010 / Accepted May 13, 2011 / Published online September 9, 2011 /

this note, we study an $h$-adaptive finite element method for solving (1.1) with a mesh dependent artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{1.2}$$

where $\varepsilon$ is suitably chosen together with an $h$-adaptivity strategy to enhance accuracy. An advantage of this approach is that the solution is regularized by an artificial viscosity and the entropy condition should be satisfied within this framework. However, one main question in this approach is whether we can solve (1.2) accurately and efficiently with a sharp viscous shock layer existing in the solution when $\varepsilon$ diminishes with the mesh size. Or equivalently, can we resolve the sharp layer while keeping the overall error small. These questions will be addressed by using an adaptive mechanism to adjust grid distribution automatically with mesh refinement in regions where small scale features (such as shock layers and vortex sheets) exist.

Adaptive finite element methods have been extensively studied and applied for solving linear parabolic partial differential equations [1,4], and also explored for nonlinear hyperbolic conservation laws [5]. This approach is efficient for solving problems whose solutions contain multiple features. One important class of adaptive strategies is based on the measurement of the residual with certain norm. The underlying mesh is adjusted locally to obtain an even distribution of the *a posterior* error or a reduction of the overall error.

The computation in this note is based on the *a posterior* error estimation of Johnson and Szepessy [5]. In their work, they provided the *a posterior* error estimates for $\varepsilon = CN^{-1}$, where $N$ is the total number of elements, in the following form

$$||e|| \le C^s C^i \left\| \frac{h^2}{\varepsilon} R(U) \right\|. \tag{1.3}$$

Here and below, the unmarked norm is the $L^2$-norm, $e = u - U$, where $u$ is the exact solution of (1.1) and $U$ is the finite element numerical solution of (1.2). $R(U) = U_t + f(U)_x - \varepsilon U_{xx}$ is the residual of the finite element solution $U$ (evaluated appropriately). $C^s$ is the stability constant and $C^i$ is the interpolation constant, which depends only on the degree of the polynomials and the shape of the finite elements. It should also be noted that $C^s$ depends on both the analytic and numerical solutions of the underlying differential equation. Therefore, strictly speaking the above estimate is not a usual *a posterior* error estimate which should only depend on $U$ and $h$. As commented in [5], $C^s$ is in general a moderate number. Analytic estimation of $C^s$ is restricted to certain model cases only, where the system is strictly hyperbolic in one dimension allowing the presence of weak shocks, noninteracting shocks and rarefaction waves. In general, it is reasonable and realistic to estimate the bound of $C^s$ computationally. We refer to [5] for the details. It is suggested in [5] that adaption of the grids could be carried out based on this *a posterior* error estimate, but it remains unclear what accuracy can be achieved by adapting the mesh based on this estimation, theoretically or numerically. Our computation in this note indicates that an order of $N^{-3/2}$ can be achieved for the two benchmark problems in scalar and system cases.

## 2. Numerical Results

The adaptive strategy here is that the mesh is adapted in an iterative way in order to get equal or close to equal amount of $||\frac{h^2}{\varepsilon}R(U)||$ on each element. A threshold is set to stop the iteration when the change of the total residue $||\frac{h^2}{\varepsilon}R(U)||$ is stagnant. The test problems here are the model cases we mentioned above, where the bound of stability constant $C^s$ could be