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ON THE CONSTRUCTION OF WELL-CONDITIONED HIERARCHICAL BASES FOR TETRAHEDRAL $\mathcal{H}(\mathbf{curl})$-CONFORMING NÉDÉLEC ELEMENT*

Jianguo Xin
Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
Email: jxin@uncc.edu
Nailong Guo
Department of Natural Sciences and Mathematics, Johnson C. Smith University, Charlotte, NC 28216, USA
Email: nguo@jcsu.edu
Wei Cai
Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
Beijing International Center for Mathematical Research, Beijing 100871, China
Email: wcai@uncc.edu

Abstract

A partially orthonormal basis is constructed with better conditioning properties for tetrahedral $\mathcal{H}(\mathbf{curl})$-conforming Nédélec elements. The shape functions are classified into several categories with respect to their topological entities on the reference 3-simplex. The basis functions in each category are constructed to achieve maximum orthogonality. The numerical study on the matrix conditioning shows that for the mass and quasi-stiffness matrices, and in a logarithmic scale the condition number grows linearly vs. order of approximation up to order three. For each order of approximation, the condition number of the quasi-stiffness matrix is about one order less than the corresponding one for the mass matrix. Also, up to order six of approximation the conditioning of the mass and quasi-stiffness matrices with the proposed basis is better than the corresponding one with the Ainsworth-Coyle basis *Internat. J. Numer. Methods. Engrg.*, 58:2103-2130, 2003. except for order four with the quasi-stiffness matrix. Moreover, with the new basis the composite matrix $\mu M + S$ has better conditioning than the Ainsworth-Coyle basis for a wide range of the parameter μ.

Mathematics subject classification: 65N30, 65F35, 65F15.
Key words: Hierarchical bases, Tetrahedral $\mathcal{H}(\mathbf{curl})$-conforming elements, Matrix conditioning.

1. Introduction

The Nédélec elements [20] are the natural choices when problems in electromagnetism are solved by the conforming finite element methods. Hierarchical bases are more convenient to use when the p-refinement technique is applied with the finite element methods [7]. Webb [28] constructed hierarchical vector bases of arbitrary order for triangular and tetrahedral finite elements. It was shown [12] that the basis functions in [28] indeed span the true Nédélec space defined in [20]. A basis in terms of the affine coordinates was also given [12]. Inspired by

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Nédélec's foundational work [20] and following Webb [28], many researchers had constructed various hierarchical bases for several commonly known elements in 2D and 3D [1,3–5,15,16,21, 25,27]. From the perspective of differential forms, Hiptmair [13] laid a general framework for canonical construction of $\mathcal{H}(\mathbf{curl})$- and $\mathcal{H}(\mathrm{div})$-conforming finite elements. In this respect, the reader is referred to the works [14,22–24] and the monograph [9].

One problem with hierarchical bases is the matrix ill-conditioning when higher-order bases are applied [2,28,31,32]. For a hierarchical basis to be useful, the issue of ill-conditioning has to be resolved. Using Gram-Schmidt orthogonalization procedure Webb [28] gave the explicit formulas of the basis functions up to order three for triangular and tetrahedral elements. Following the same line of development [28], i.e., decomposing the basis functions into rotational and irrotational groups, Sun and collaborators [27] investigated the conditioning issue more carefully and also gave the basis functions up to the third order. Ainsworth and Coyle [3] studied both the dispersive and conditioning issues for the hierarchical basis on the hybrid quadrilateral/triangular meshes. With the aid of Jacobi polynomials, the interior bubble functions are made orthogonal over an equilateral reference triangle [3]. With this partial orthogonality it was shown that the condition numbers of both the mass matrix and the stiffness matrix could be reduced significantly [3]. Using Legendre polynomials Jørgensen et al. constructed a near-orthogonal basis for the quadrilaterals and suggested that the same procedure could be applied for the triangles with the help of collapsed coordinate system [17]. Partially addressing the conditioning issue, Schöberl and Zaglmayr [25] created bases for high-order Nédélec elements with the property of local complete sequence. The key components in their construction [25] are using (i) the gradients of scalar basis functions and, (ii) scaled and integrated Legendre polynomials. However, the ill-conditioning issue was pronounced with higher-order approximation and moderate growth of the condition number was reported [25]. A new hierarchical basis with uncommon orthogonality properties was constructed by Ingelström [15] for tetrahedral meshes. It is shown that higher-order basis functions vanished if they were projected onto the relatively lower-order $\mathcal{H}(\mathbf{curl})$-conforming spaces [15] using the Nédélec interpolation operator [20], and [15] such a basis was well suited for multi-level solvers. Using the orthogonalization procedure by Shreshevskii [26] and conforming to the Nédélec [20] condition, Abdul-Rahman and Kasper proposed a new hierarchical basis for the tetrahedral element [1].

The Gram-Schmidt scheme used by Webb [28] or the orthogonalization method applied by Abdul-Rahman and Kasper [1] involves a linear system of equations to be solved, and the coefficients associated with the basis functions in general cannot be expressed in closed forms. The focus of the current work is to investigate the possibility of constructing a well-conditioned hierarchical basis for the tetrahedral $\mathcal{H}(\mathbf{curl})$-conforming elements without recourse to the Gram-Schmidt orthogonalization. The construction is made possible by the results of orthogonal polynomials of several variables on an n-simplex [11]. The basis functions of any approximation order are given explicitly in closed form. Our work is based upon the studies by Ainsworth and Coyle [5], and by Schöberl and Zaglmayr [25]. The main goal of this study is to try to resolve the conditioning issue or at least partially, which was missed in the study by Ainsworth and Coyle [5].

The rest of this paper is organized as follows. The construction of basis functions is given in Section 2. Numerical results of matrix conditioning and sparsity are shown in Section 3. Concluding remarks are included in Section 4.