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SUPERCONVERGENCE OF A DISCONTINUOUS GALERKIN METHOD FOR FIRST-ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS*

Dongfang Li and Chengjian Zhang
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
Email: cjzhang@mail.hust.edu.cn, lidongfang1983@gmail.com

Abstract

This paper deals with the discontinuous Galerkin (DG) methods for delay differential equations. By an orthogonal analysis in each element, the superconvergence results of the methods are derived at nodal points and eigenpoints. Numerical experiments will be carried our to verify the effectiveness and the theoretical results of the proposed methods.

Mathematics subject classification: 65N12, 65N30.
Key words: Discontinuous Galerkin methods, Delay differential equations, Orthogonal analysis, Superconvergence.

1. Introduction

Delay differential equations frequently arise in a fast variety of scientific problems, such as relativistic dynamics, nuclear reactor, neural network, electric circuit and viscoelasticity mechanics, see, e.g., [11, 14]. The last several decades have witnessed a fast development in computational implementation and numerical analysis for various delay differential equations, see, the monographs e.g., [2, 3] and the references therein. It is noted that most authors have employed finite difference methods, such as linear multistep methods, one-leg methods, Runge-Kutta methods and general linear methods, see, e.g., [19, 20].

Besides the finite difference methods, it is well-known that the finite element methods are also a class of effective numerical methods for solving differential equations, and usually some superconvergence results are available, see, e.g., [1, 4–6, 9, 13, 15–17]. Up to now, however, there have been very few papers on finite elements for solving delay differential equations (DDEs). Generally speaking, solution behaviors of the DDEs are more complicated than those for the standard differential equations since the former depends not only on the present but also on the history. The presence of a delay term could change a system's dynamic properties such as stability, oscillation, bifurcation, chaos and etc. In [10, 18] continuous Galerkin finite element (CGFE) methods are applied to DDEs with one-delay and multi-delay, are respectively, and a number of superconvergence results of the CGFE methods are obtained. In [12], continuous and discrete finite element approximations to a class of parabolic delay differential equations are investigated optimal error estimates in L^2, H^1 and L^∞ norms are obtained.

As an important subclass of finite element methods, the discontinuous Galerkin (DG) methods have been found very useful in scientific engineering. For detail description of the method as well as its development, we refer the readers to the review paper [7] and the special issue of

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Journal of Scientific Computing [8]. Up to now, the DG methods have been proved locally conservative, stable, and high-order accurate. Since the DG methods have many desired properties, it will be interesting to apply such methods to DDEs. In [15], we proved that the DG methods have the ability to preserve stability of the underlying systems. Following our earlier work, the superconvergence results of the methods will be derived at nodal points and eigenpoints in this work.

The rest of the paper is structured as follows. In Section 2, we introduce a class of DG methods for DDEs. In Section 3, we analyze errors of the methods and prove that the DG methods have superconvergence at nodal points and eigenpoints. In Section 4, numerical experiments will be used to confirm the effectiveness and the theoretical results of the methods. Finally, conclusions and discussions for this paper are summarized in Section 5.

2. Delay Differential Equations and Their DG Methods

In this section, we will give a discretization scheme based on the DG methods for a class of linear DDEs. Consider the following DDEs with delay $\tau > 0$:

$$\begin{cases} u'(t) + a(t)u(t) + b(t)u(t-\tau) = f(t), & t_0 \le t \le T, \\ u(t) = \psi(t), & t_0 - \tau \le t \le t_0, \end{cases} \tag{2.1}$$

where the functions $a(t), b(t), f(t), \psi(t)$ are assumed to be continuous on their respective domains so that the above delay system has a unique solution $u \in H^1([t_0, +\infty))$. When $\tau \ge T - t_0$, system (2.1) becomes a linear ordinary differential equation (ODE)

$$\begin{cases} u'(t) + a(t)u(t) = -b(t)\psi(t-\tau) + f(t), & t_0 \le t \le T, \\ u(t_0) = \psi(t_0). \end{cases} \tag{2.2}$$

Such an ODE system has been solved by CGFE methods and DG methods in many references, see, e.g. [1,5,7,9,13,17]. In this work, we always assume $\tau \le T - t_0$ so that the non-degenerate delay systems can be considered.

For the discretization of system (2.1) by a class of DG methods, we divide the interval $[t_0, T]$ with a uniform mesh:

$$\mathcal{J}^h: \quad t_0 < t_1 < \cdots < t_N,$$

where $t_n = t_0 + 2nh, h = \tau/(2k)$, k is a given positive integer and the maximum index N satisfies $t_{N-1} < T \le t_N$. Moreover, we write that the element $J_n = (t_{n-1}, t_n]$, the half-integer node $t_{n-1/2} = (t_n + t_{n-1})/2 (= t_0 + (2n-1)h)$ and the extended interval $J = [t_0, t_N]$, and define m-degree discontinuous finite element space as follows:

$$S^h = \big\{ v : \ v\big|_{J_n} \in P_m(J_n), n = 1, 2, \cdots, N \big\},$$

where $P_m(J_n)$ denotes the set of all polynomials of degree $\le m$ on J_n. Note that when a function $U \in S^h$, it implies that U is allowed to be discontinuous but left-continuous at the nodal points t_n, i.e. $U(t_n) = U(t_n - 0)$. For brevity, we introduce the following notations:

$$U_n^- = U(t_n - 0) = U(t_n) = U_n, \quad U_n^+ = U(t_n + 0), \quad [U_n] = U_n^+ - U_n^-.$$

In general case, the span $[U_n] \ne 0$. As there is no request that $U \in S_h$ is continuous at the nodal points, U has $(m+1)$-degree of freedom on an element J_n.