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FINITE ELEMENT ANALYSIS OF OPTIMAL CONTROL PROBLEM GOVERNED BY STOKES EQUATIONS WITH L^2-NORM STATE-CONSTRAINTS*

Haifeng Niu
School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China
School of Mathematics, Shandong University, Jinan 250100, China
Email: niuhaifeng@mail.sdu.edu.cn
Danping Yang
Department of Mathematics, East China Normal University, Shanghai 200241, China
Email: dpyang@math.ecnu.edu.cn

Abstract

An optimal control problem governed by the Stokes equations with L^2-norm state constraints is studied. Finite element approximation is constructed. The optimality conditions of both the exact and discretized problems are discussed, and the *a priori* error estimates of the optimal order accuracy in L^2-norm and H^1-norm are given. Some numerical experiments are presented to verify the theoretical results.

Mathematics subject classification: 49J20, 65N30
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1. Introduction

In many engineering applications, the control problems of various flow are very important. One can find lots of useful models for optimal control problems of flow motion with purposes of achieving some desired objectives in real-life applications. Many of those problems come from the fluids flow, aeronautical, chemical engineering, magnetic field and heat sources using radiation or the laser technology, see, for instance, [14, 15, 19, 21, 22, 31] and the references cited therein. There have been extensive research carried out on various theoretical aspects of optimal control problems governed by flow, for example, see [1, 15–18, 24], where control-constrained problems are studied. The state constrained control problems are also frequently met in practical applications, which have aroused many researchers' interests, for example, see [6, 7, 11, 35] for state constrained elliptic control problems. Besides the pointwise state constrained cases as in the above references [6, 11], the integral or the energy of the state are worth concerning in many control problems. For example, one probably wishes to constrain the concentration, the temperature in the average sense in some domain, or the kinetic energy of the flow, etc. In [7], Casas discussed the numerical approximation of optimal control problems governed by a second order semi-linear elliptic partial differential equation associated with finitely many state constraints and gave *a priori* error estimates in H^1-norm. In [25], Liu, Yang and Yuan studied the integral state-constrained control problems governed by an elliptic PDE, proposed a gradient projection algorithm and derived the *a priori* error estimates of the

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optimal order accuracy in L^2- and L^∞-norms. Furthermore, Yuan and Yang analyzed the finite element approximation of L^2-norm state-constrained elliptic control problems and constructed the Uzawa type iterative method in [35]. However, up to now, there has no systematical analysis in the literature for optimal control problems governed by the Stokes equations with state constraints. It is more complicated to study the finite element approximation of the flow control since one has to handle the mixed element.

The purpose of this article is to study the optimal control problems governed by the Stokes equations with L^2-norm constraints for the velocity, where the control is distributed in Ω without constraint. We construct the finite element approximation and analyze optimality conditions for both the exact and the discretized problems. We study *a priori* error estimates between the exact solution and its finite element approximation in L^2-norm and H^1-norm.

The outline of the article is as follows. In Section 2, we state the model problem and construct its finite element approximation. In Section 3, we derive the *a priori* error estimates for the finite element approximation. Finally, in Section 4, we give the Arrow-Hurwicz algorithm and perform some numerical experiments to verify the theoretical results given in Section 3.

2. Control Problem and Finite Element Approximation

Throughout the article, we use the standard definitions and notations of the Sobolev spaces as in [2]. Let Ω be a bounded and open connected domains in $\mathbb{R}^d$ for $d = 2$ or 3. Denote by $\mathbf{v} = (v_1, \cdots, v_d)$ the d-dimensional vector-valued function, $\mathbf{L}^p(\Omega) = \big(L^p(\Omega)\big)^d$, $\mathbf{H}^m(\Omega) = \big(H^m(\Omega)\big)^d$ and $\mathbf{W}^{m,p}(\Omega) = \big(W^{m,p}(\Omega)\big)^d$ the usual vector-valued Sobolev spaces with norms $\|\cdot\|_{m;\Omega} = \|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{m,p;\Omega} = \|\cdot\|_{W^{m,p}(\Omega)}$, respectively. We use $(\cdot,\cdot)_G$ to denote the inner product defined on the bounded and open set G, and if the $G = \Omega$ we omit the subscript, e.g., $(\cdot,\cdot)$. Introduce some function spaces

$$\mathbf{U} = \mathbf{L}^2(\Omega), \quad \mathbf{H} = \big(H_0^1(\Omega)\big)^d, \quad Q = \Big\{q \in L^2(\Omega);\ \int_\Omega q = 0\Big\},$$

which stand for the control space, the velocity sate space and the pressure state space, respectively.

2.1. Optimal control problem

We first state the model problem and its weak form. Let α be a positive constant and the objective functional $\mathcal{J} : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ be defined as:

$$\mathcal{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\int_\Omega |\mathbf{y} - \mathbf{y}_d|^2 + \frac{\alpha}{2}\int_\Omega |\mathbf{u} - \mathbf{u}_0|^2.$$

For a positive integer M, the constraint set is given by $\mathbf{K} = \bigcap_{i=1}^{M} \mathbf{K}_i$, where

$$\mathbf{K}_i = \Big\{\mathbf{w} \in \mathbf{L}^2(\Omega);\ \|\mathbf{w}\|_{0;\Omega_i} \le \gamma_i\Big\}, \quad 1 \le i \le M, \tag{2.1}$$

and $\{\Omega_i\}_{i=1}^M$ are nonempty subsets of Ω such that $\Omega_j \bigcap \Omega_k = \emptyset$ for all $1 \le j < k \le M$, and the real number γ_i satisfies $\gamma_i > 0$ for all $1 \le i \le M$.

We investigate the following state-constrained optimal control problem:

$$\min_{\mathbf{y}(\mathbf{u})\in\mathbf{K}} \mathcal{J}(\mathbf{y}(\mathbf{u}), \mathbf{u}) \tag{2.2}$$