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ON EXTRAPOLATION CASCADIC MULTIGRID METHOD*

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Abstract

Based on an asymptotic expansion of (bi)linear finite elements, a new extrapolation formula and extrapolation cascadic multigrid method (EXCMG) are proposed. The key ingredients of the proposed methods are some new extrapolations and quadratic interpolations, which are used to provide better initial values on the refined grid. In the case of triple grids, the errors of the new initial values are analyzed in detail. The numerical experiments show that EXCMG has higher accuracy and efficiency.

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Key words: Cascadic multigrid, Finite element, New extrapolation, Error analysis.

1. Introduction

To solve a linear system of equations derived by the finite difference method or finite element method for PDEs, it is expected that the solution of an N-order system can be obtained by using $\mathcal{O}(N)$. Multigrid method (MG) was first realized this purpose and was then become one of the most effective algorithms. There are two important types of MG methods:

(i) (Classical) Multigrid Method. The basic idea of MG was early proposed by Fedorenko [1] in 1964, and was re-discovered by Brandt [2] in 1977. Later on, MG has been gradually completed by Bank-Dupont, Braess-Hackbush, McCormick, Bramble-Pasciak-Xu et al., see, e.g., [3,4]. In MG, three operators between different levels of grid, i.e., the interpolation, restriction and iteration, are used. There are V-cycle and W-cycle algorithms.

(ii) Cascadic Multigrid Method (CMG). CMG was proposed by Borneman-Deunfhard [5] in 1996 and Shaidurov [6, 7] (also in 1996, called Cascadic CG; 1999, discussed the domain with re-entrant corners). Shi and Xu et al. [8–10] made further analysis and extensions in 1998 and 1999. Later on, CMG has been generalized to the nonconforming elements, finite volume method, nonlinear and parabolic problems and so on, see, e.g., [11–19]. In CMG, only the interpolation and iteration from coarse grids to refined grids are used. Compare with the classical CG method, the code for CMG is simpler.

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Let Ω be a planar polygon with the boundary Γ, we consider an elliptic problem to find $u \in H_0^1$ such that

$$A(u,v) \equiv \int_\Omega \nabla u \nabla v dx = (f,v), \quad \forall v \in H_0^1 = \left\{ u: \ u \in H^1(\Omega), \ u = 0 \ on \ \Gamma \right\}, \tag{1.1}$$

where the bilinear form $A(u,v)$ is bounded and H_0^1-coercive, $A(u,u) \geq \nu ||u||_1^2$.

Subdivide Ω into a sequence of triangular or rectangular grids $Z_l, l = 0,1,2,\cdots,L$ with step-length $h_l = h_0/2^l$. Denote by $V_l \subset H_0^1$ the (bi)linear finite element subspace on the grid Z_l and by $U^l \in V_l$ the corresponding finite element solution satisfying

$$A(U^l, v) = (f,v), \quad v \in V_l, \quad l = 0,1,\cdots,L, \tag{1.2}$$

which leads to a linear system of equation

$$K_l U^l = b_l, \quad on \ \ Z_l, \quad l = 0,1,\cdots,L. \tag{1.3}$$

Define the linear interpolation $u^l = I_1 u$ of u, and let the error $e^l = U^l - u^l$ on Z_l. The energy error is defined by $||e^l||_{K_l} = (K_l e^l, e^l)^{1/2}$. It is known that the norms $||e^l||_{K_l}$ and $||e^l||_1$ are equivalent.

Algorithm 1.1 Assume that the exact solution $\bar{U}^0$ on Z_0 given, CMG consists of three steps, $l = 1,\cdots,L$:
Step 1. Take the linear interpolation $I_1\bar{U}^{l-1}$ to define the initial value $U^{l,0} = I_1\bar{U}^{l-1}$ on Z_l;
Step 2. Use the operator S_l to get the iteration solution $\bar{U}^l = S_l^{m_l} U^{l,0}$;
Step 3. Come back to steps 1 and 2 if $l < L$, until get the final solution $\bar{U}^L$ on Z_L.

In CMG, the errors are often measured by K_l-norm, $||u - U^l||_{K_l} = \mathcal{O}(h_l)$. Since the gradient $D(U^l - u^l) = \mathcal{O}(h_l^2)$ is superconvergent on the (piecewise) uniform grid, the iteration error

$$||u - U^l||_{K_l} \leq ||u - u^l||_{K_l} + ||u^l - U^l||_{K_l} = \mathcal{O}(h_l)$$

is easily attained. Thus CGM is efficient in the energy norm and applicable to many problems, for example, the elastic system.

However, CMG is not of the optimal convergence in L^2-norm. By the embedding theorem, we can only get

$$||u - U^l||_0 \leq C||u - U^l||_1 = \mathcal{O}(h_l),$$

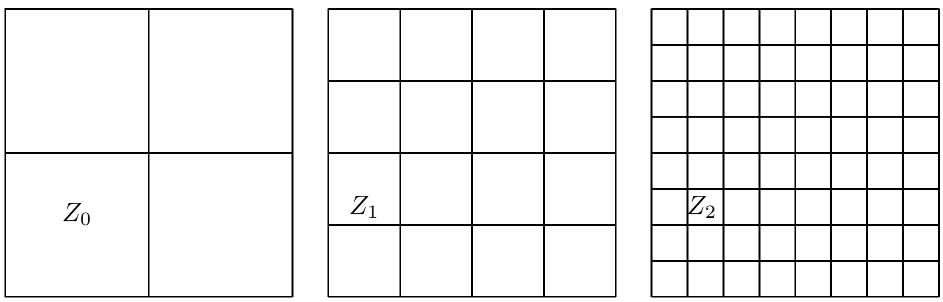

Fig. 1.1. Triple grids: $K_0U^0 = b_0$ *in* Z_0; $K_1U^1 = b_1$ *in* Z_1; $K_2U^2 = b_2$ *in* Z_2.