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ESTIMATION FOR SOLUTIONS OF ILL-POSED CAUCHY PROBLEMS OF DIFFERENTIAL EQUATION WITH PSEUDO-DIFFERENTIAL OPERATORS*

Prat II. Case of Second Order Operators

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Abstract

The estimation for solutions of the ill-posed Cauchy problems of the differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\ \forall t\in(0,1)$$

is discussed, where $A(t)$ is a 2-nd order p. d. o. and $N(t)$ is a uniformly bounded $H\to H$ linear operator. Two estimates of $\|u(t)\|$ are obtained.

This part is a continuation of Part I, so we use the same notation as in Part I and continue the section numbers. In section 4 we deal with diagonalizors of r-th order p. d. o.. Then in section 5 we derive the desired estimates.

§ 4. Diagonalizors

We will construct diagonalizors for r-th order p. d. o. in this section.

Lemma 4.1. *Let $P(t)\in\mathscr{L}^0_{s_1}$ have the symbol $p(t,x,\xi)\in\{S^0_1\}$ satisfying*

$$|\det p(t,x,\xi)|\geqslant \text{const}\ \gamma>0,\ \forall(t,x,\xi)\in([0,1]\times R^m\times(|\xi|-1)).$$

Then for any integer j, there exists $P^{(-1)}_j(t)\in\mathscr{L}^0_{s_1}$ such that

$$P(t)P^{(-1)}_j(t)-I=N_{R(-j)}(t)\in\mathscr{L}^{-j},$$
$$P^{(-1)}_j(t)P(t)-I=N_{L(-j)}(t)\in\mathscr{L}^{-j}. \tag{4.1}$$

Proof. Let $P^{(-1)}_j(t)$ have the symbol

$$b(t,x,\xi)=\sum_{l=0}^{j-1}b_l(t,x,\xi)|\xi|^{-l}$$

with undetermined coefficients $b_l(t,x,\xi)$. We have to determine these coefficients such that $N_{R(-j)}(t)$ and $N_{L(-j)}(t)$ belong to $\mathscr{L}^{-j}$.

Applying Theorem 1.3 for $P(t)$ and $P^{(-1)}_j(t)$, we have that the difference between $P(t)P^{(-1)}_j(t)-I$ and the p. d. o. $N^a_{R(-j)}(t)$ with the symbol

$$\sum_{k=0}^{j-1}\Big(\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha p(t,x,\xi)\partial^\alpha b(t,x,\xi)\Big)-I_{n\times n}$$

belongs to $\mathscr{L}^{-j}$. Calculating the coefficients of $|\xi|^{-i}$ for $i=0,1,\cdots,j-1$ in the

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symbol of $N^a_{R(-j)}(t)$ and letting them equal to $\theta_{n\times n}$, we obtain.

$$b_0(t, x, \xi)=p^{-1}(t, x, \xi)$$

$$\cdots\cdots$$

$$b_l(t, x, \xi)=-p^{-1}(t, x, \xi)\sum_{k=1}^{l}(-1)^k\sum_{|\alpha|=k}\frac{1}{\alpha!}D^\alpha p(t, x, \xi)\partial^\alpha b_{l-k}(t, x, \xi),$$

$$\cdots\cdots$$

Since $p(t, x, \xi)\in\{S_1^0\}$ and $|\det p(t, x, \xi)|\geqslant\gamma>0$, we have $p^{-1}(t, x, \xi)\in\{S_1^0\}$ and $b(t, x, \xi)\in\{S_1\}$. Consequently the p. d. o. $P_j^{(-1)}(t)$ obtained above belongs to $\mathscr{L}^0_{s_1}$ and $N^a_{R(j)}(t)$ has the symbol whose first term with nonzero coefficient is of $|\xi|^{-j}$; hence from Theorem 1.1 $N^a_{R(j)}(t)\in\mathscr{L}^{-j}_{s_1}$. Finally we get

$$N_{R(-j)}(t)=[(P(t)P_j^{(-1)}(t)-I)-N^a_{R(-j)}(t)]+N^a_{R(-j)}(t)\in\mathscr{L}^{-j}.$$

Similarly we can construct the p. d. o. $\hat{P}_j^{(-1)}(t)\in\mathscr{L}^0_{s_1}$ such that

$$\hat{N}_{L(-j)}(t)=\hat{P}_j^{(-1)}P(t)-I\in\mathscr{L}^{-j}.$$

Multiplying the relation $P(t)P_j^{(-1)}(t)P(t)=(I+N_{R(-j)}(t))P(t)$ by $\hat{P}_j^{(-1)}(t)$ on the left we obtain

$$\begin{aligned}(I+\hat{N}_{L(-j)}(t))P_j^{(-1)}(t)P(t)&=\hat{P}_j^{(-1)}(t)(I+N_{R(-j)}(t))P(t)\\&=(I+\hat{N}_{L(-j)}(t))+\hat{P}_j^{(-1)}(t)N_{R(-j)}(t)P(t).\end{aligned}$$

Hence

$$\begin{aligned}N_{L(-j)}(t)&=P_j^{(-1)}(t)P(t)-I\\&=\hat{N}_{L(-j)}(t)+\hat{P}_j^{(-1)}(t)N_{R(-j)}(t)P(t)-\hat{N}_{L(-j)}(t)P_j^{(-1)}(t)P(t)\in\mathscr{L}^{-j}.\end{aligned}$$

The lemma is proved.

Let $a_0(t, x, \xi)$ be an $(n\times n)$ matrix belonging to $\{S_1^0\}$ and positive homogeneous of degree zero in ξ. As in Part I, $a_0(t, x, \xi)$ is said to be uniformly diagonalizable, if the $(n\times n)$ matrix consisting of the eigenvectors of $a_0(t, x, \xi)$ is uniformly nonsingular and belongs to $\{S_1^0\}$. Uniform nonsingularity means

$$|\det p(t, x, \xi)|\geqslant\text{const }\gamma>0,\ \forall(t, x, \xi)\in D=([0, 1]\times R^m\times(|\xi|=1)).$$

Obviously

$$p(t, x, \xi)a_0(t, x, \xi)=J_0(t, x, \xi)p(t, x, \xi), \tag{4.2}$$

where $J_0(t,x,\xi)\in\{DS_1^0\}$ is a diagonal matrix consisting of the eigenvalues $\lambda_j(t,x,\xi)$ of $a_0(t, x, \xi)$.

Lemma 4.2. *Let $A(t)\in\mathscr{L}^r_{s_1}$ have the symbol $a(t, x, \xi)=\sum_{l=0}^{N}a_l(t, x, \xi)|\xi|^{r-l}$, the first coefficient $a_0(t, x, \xi)$ of which is uniformly diagonalizable. If the eigenvalues $\lambda_j(t, x, \xi)$ of $a_0(t, x, \xi)$ satisfy*

$$|\lambda_l(t, x, \xi)-\lambda_j(t, x, \xi)|\geqslant\text{const }\gamma_2>0,\quad\forall(t, x, \xi)\in D\text{ and }l\neq j, \tag{4.3}$$

then there exists a p. d. o. *$D(t)\in\mathscr{L}^0_{s_1}$ such that*

$$D(t)P(t)A(t)=E(t)D(t)P(t)+N_0(t)\quad\text{in }\mathscr{L}, \tag{4.4}$$

where $P(t)\in\mathscr{L}^0_{s_1}$ has the symbol $p(t, x, \xi)$ consisting of the left eigenvectors of $a_0(t, x, \xi)$, $E(t)\in\mathscr{L}^r_{Ds_1}$ has the diagonal symbol

$$J(t, x, \xi)=\sum_{l=0}^{r-1}J_l(t, x, \xi)|\xi|^{r-l}, \tag{4.5}$$

in which $J_0(t, x, \xi)\in\{DS_1^0\}$ is the diagonal matrix consisting of the eigenvalues of