CARDINALITIES OF RESTRICTED RANGES*

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Abstract

Let l and u be upper and lower semicontinuous extended functions on $[a, b]$, respectively, with $l\leqslant u$. Let H be an n-dimensional Haar subspace and $K=\{p\in H:l\leqslant p\leqslant u\}$. This paper gives complete characterizations of K satisfying

$$\operatorname{card} K=0 \text{ or } 1 \text{ or } \infty$$

under certain assumptions, where card K denotes the cardinality of K.

1. Introduction

In approximation by polynomials having restricted ranges[1] and in simultaneous approximation[2] the following problem may be proposed:

Let l and u be upper and lower semicontinuous functions on $X\equiv[a, b]$ (which may take $-\infty$ and $+\infty$, but $l<+\infty$ and $u>-\infty$), respectively, with $l\leqslant u$. Let H be an n-dimensional subspace of $C(X)$ and $K=\{p\in H:l\leqslant p\leqslant u\}$. Characterize K such that

$$\operatorname{card} K=0 \text{ or } 1 \text{ or } \infty,$$

where card K denotes the cardinality of K.

In this paper we give an answer to this problem for H being a Haar subspace. In detail, we give complete characterizations of K satisfying card $K=0$ or 1 or ∞ under certain assumptions.

To begin with let us introduce the following notation.

For $p\in H$ denote

$$X_p^l=\{x\in X:p(x)\leqslant l(x)\},$$
$$X_p^u=\{x\in X:p(x)\geqslant u(x)\},$$
$$X_p=X_p^l\cup X_p^u,$$
$$\sigma(x)=\begin{cases}1, & x\in X_p^l\\ -1, & x\in X_p^u.\end{cases}$$

By definition if $p(x)=l(x)=u(x)$, $\sigma(x)$ may take both 1 and -1.

A system of $n+1$ ordered points

$$x_1<x_2<\cdots<x_{n+1} \tag{1}$$

in X_p is said to be an alternation system of p (with respect to (l, u)) if it satisfies

$$\sigma(x_{i+1})=-\sigma(x_i), \quad i=1, \cdots, n. \tag{2}$$

It should be pointed out that the restrictions on l and u being upper and lower

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semicontinuous are trivial, because for any l and u we can assume

$$\bar{l}(x)=\limsup_{y\to x} l(y), \qquad \bar{u}(x)=\liminf_{y\to x} u(y)$$

instead, which are upper and lower semicontinuous, respectively[5], and satisfy that

$$\text{card } K=\text{card}\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\},$$

To verify the last equality we note that, on the one hand, from $l\leqslant\bar{l}\leqslant\bar{u}\leqslant u$ card $K\geqslant$card$\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\}$ follows, and on the other hand, $l\leqslant p\leqslant u$ implies

$$\limsup_{y\to x} l(y)\leqslant\limsup_{y\to x} p(y)=\liminf_{y\to x} p(y)\leqslant\liminf_{y\to x} u(y),$$

namely, $\bar{l}(x)\leqslant p(x)\leqslant\bar{u}(x)$, from which card $K\leqslant$card$\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\}$ follows.

2. Main Theorems

Theorem 1. *Let $l<u$ and let H be an n-dimensional Haar subspace. Then for $p\in K$ the following statements are equivalent each to other:*

(a) $K=\{p\}$, *i.e.*, card $K=1$;

(b) $\max_{x\in X_p}\sigma(x)q(x)\geqslant 0,\ \forall q\in H$;

(c) $\max_{x\in X_p}\sigma(x)q(x)>0,\ \forall q\in H\setminus\{0\}$;

(d) $0\in\mathscr{H}\{\sigma(x)\hat{x}:x\in X_p\}$, *where $\mathscr{H}$ denotes the convex hull* [4, p. 17] *and* $\hat{x}=(\phi_1(x),\cdots,\phi_n(x))$ *with $\phi_1,\cdots,\phi_n$ being a basis in H*;

(e) *p possesses an alternation system with respect to (l, u).*

Proof. (a)⇒(b). Suppose not and let q satisfy $\max_{x\in X_p}\sigma(x)q(x)<0$, i. e., $\sigma(x)q(x)<0,\ \forall x\in X_p$. We are to prove that $r_t=p-tq$ satisfies $l<r_t<u$ for some $t>0$. Hence from $r_t\neq p$ a contradiction occurs.

Let $h=\frac{1}{2}\min_{x\in X}\{u(x)-l(x)\}(>0)$ and $e=\max_{x\in X}|q(x)|$. Denote

$$Y_1=\{x\in X:p(x)-l(x)>h \text{ and } q(x)>0\},$$
$$Y_2=\{x\in X:u(x)-p(x)>h \text{ and } q(x)<0\},$$
$$Y=X\setminus(Y_1\cup Y_2).$$

Taking $t_1=h/e$, we have that for $x\in Y_1$ and $0<t\leqslant t_1$

$$r_t(x)=p(x)-tq(x)>l(x)+h-t_1e=l(x)$$

and

$$r_t(x)=p(x)-tq(x)\leqslant u(x)-tq(x)<u(x),$$

that is,

$$l(x)<r_t(x)<u(x). \tag{3}$$

Similarly, (3) holds for $x\in Y_2$ and $0<t\leqslant t_1$.

On the other hand, it is easy to see that $X_p\subset Y_1\cup Y_2$ and, hence,

$$l(x)<p(x)<u(x),\quad \forall x\in Y.$$

Since Y is compact, we can find a number $t_2>0$ so that (3) is also valid for all $x\in Y$ and $0<t\leqslant t_2$.

Thus $l<r_t<u$ is valid for $t=\min\{t_1, t_2\}$.

(b)⇒(d). (b) implies that the linear inequalities