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# LACUNARY SPLINE INTERPOLATION*

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**Abstract**

In this paper, some general kinds of lacunary spline interpolations are discussed. Existence and uniqueness are proved. Order of convergence and saturation of approximation are also obtained. The results generalize some individual results in references.

## 1. Introduction and Notation

A kind of quintic lacunary splines was studied in [1], and some further information was given by [2], [3]. Since then, various lacunary splines with different degrees have appeared in this background (c. f. [4], [5], [6]). Among them, a kind of higher degree lacunary splines discussed in [6] can be considered as an extension of Meir and Sharma's quintic spline in [1]. By a general method, the existence, uniqueness and convergence of another kind of lacunary splines were obtained in [7]. In this paper, we continue the work in this respect and discuss a more general kind of lacunary splines. The results here, in special cases, may reduce to the results of [1], [5], [6].

Through this paper, we consider interval $[0, 1]$ with a partition of the form $\Delta_N$: $0=x_0<x_1<\cdots<x_n=1$. Let $A$, $B$, $C$ denote the sets of nonnegative integers and $|A|$ be the cardinality of $A$. $P_n$ denotes the set of algebraic polynomials of degree less than or equal to $n$. $S_p(n, m, \Delta_N)\equiv\{s(x)\,|\,s(x)\in C^{m-1}[0, 1]$ and $s(x)\in P_n$ in the mesh interval $[x_i, x_{i+1}]$ of $\Delta_N\}$. The norm $\|\cdot\|$ is due to Tchebycheff. If $g$ is a function whose $j$th derivative exists at all points of $[0, 1]$ except finite number of points $y_1, y_2, \cdots, y_m$, we regard $\|g^j\|$ as $\|g^j\|\equiv\sup\limits_{1<i<m-1}\|g^j\|_{(y_i, y_{i+1})}$. Here and afterwards we always use $g^j(x)$ instead of $g^{(j)}(x)$. By saying there exists a relation $J_1\prec J_2$ between two sets of nonnegative integers $J_1\equiv\{j_1^1, j_2^1, \cdots, j_m^1\}$ and $J_2\equiv\{j_1^2, j_2^2, \cdots, j_m^2\}$, we mean that $j_i^1\leqslant j_i^2$ for all $i$.

## 2. Existence and Uniqueness

First we introduce a result on polynomial Hermite-Birkhoff interpolation (c. f. [8]):

**Definition 1.** *A $2\times m$ matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{m-1}$ consisting of zeros and ones is a two-point poised H-B matrix, if the only polynomial $P$ of degree $m-1$ satisfying $P^j(i)=0$ for all $(i, j)$ such that $e_{ij}=1$ is $P\equiv0$.*

**Lemma 1** (c. f. [8]). *A $2\times m$ matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{m-1}$ consisting of zeros and ones is a two-point poised H-B matrix if and only if the Pólya conditions*

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$$\sum_{i=0}^{1}\sum_{j=0}^{p} e_{ij} \equiv M_p \geqslant p+1, \quad 0 \leqslant p \leqslant m-1 \tag{1}$$

*are satisfied.*

For the convenience of the discussion below, we give:

**Definition 2.** *A set of nonnegative integers $J=\{j_0, j_1, \cdots, j_{m-1}\}$, $j_0<j_1<\cdots<j_{m-1}$, is said to be a symmetric set of order $m$, if $J$ corresponds to a two-point H–B matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{2m-1}$, with*

$$e_{ij}=\begin{cases}1, & j\in J,\ 0\leqslant i\leqslant 1,\\ 0, & \text{otherwise}.\end{cases}$$

*When the corresponding H–B matrix of $J$ is poised, $J$ is also called poised.*

In [7], the poised symmetric set is called symmetric Pólya set.

We notice that by this definition,

$$J^* = \{0, 2, 4, \cdots, 2m-2\} \tag{2}$$

corresponds to a $H$–$B$ matrix

$$E^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0\cdots \\ 1 & 0 & 1 & 0 & 1 & 0\cdots \end{pmatrix},$$

Obviously it satisfies the Pólya conditions (1), and none of the columns consisting of ones in $E^*$ can move from left to right to maintain Pólya conditions. So we get Lemma 2 by Lemma 1 at once:

**Lemma 2.** *The symmetric set $J=\{j_0, j_1, \cdots, j_{m-1}\}$ of order $m$ is a symmetric Pólya set if and only if*

$$J \prec J^*. \tag{3}$$

When $J$ is a symmetric Pólya set of order $n+1$, there exists a unique $p\in P_{2n+1}$ such that $p^j(i)=a_{ij}$, $j\in J$, $i=0, 1$, for given arbitrary real data $\{a_{ij}: j\in J, i=0, 1\}$.

Obviously it follows that $0\in J$ by Lemma 2 and there are unique polynomials $g_j, h_j\in P_{2n+1}$ satisfying

$$g_j^i(0)=h_j^i(1)=\delta_{ij}, \quad g_j^i(1)=h_j^i(0)=0, \quad i, j\in J, \tag{4}$$

with $J$ a symmetric Pólya set, and any $p\in P_{2n+1}$ has the unique expansion

$$p(x)=\sum_{j\in J}\{p^j(0)g_j(x)+p^j(1)h_j(x)\}.$$

In general, for any interval $[x_i, x_{i+1}]$, with $h_{i+1}=x_{i+1}-x_i>0$, we have

$$p^r(x)=\sum_{j\in J} h_{i+1}^{j-r}\left\{p^j(x_i)g_j^r\left(\frac{x-x_i}{h_{i+1}}\right)+p^j(x_{i+1})h_j^r\left(\frac{x-x_i}{h_{i+1}}\right)\right\}. \tag{5}$$

The expressions of $g_j(x)$ and $h_j(x)$ can be obtained from (4); for some special cases, this has been done (c.f. [9] for $J=J^*$).

Now, let $\Delta_N$ be a given partition of $[a, b]$, and $A$, $B$, $C$ are the sets of nonnegative integers satisfying conditions:

(i) $A$, $B$ and $C$ are disjoint from each other

$$|A|=r_0+2, \ |B|=n-r_0-1, \ |C|=2|B|,$$

(ii) $A\cup B$ is a symmetric Pólya set of order $n+1$,

(iii) $A\cup C=\{0, 1, 2, \cdots, 2n-r_0-1\}$, $B\subset\{2n-r_0, 2n-r_0+1, \cdots, 2n\}$.

We consider the interpolating lacunary spline $s\equiv S(f, A, B, C, \Delta_N)$ satisfying: