ORDER INTERVAL TEST AND ITERATIVE METHOD FOR NONLINEAR SYSTEMS*

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Abstract

An order interval test for existence and uniqueness of solutions to a nonlinear system is given. It combines the interval technique and the monotone iterative technique. It has the main merits of interval iterative methods but need not use interval arithmetic. An order interval Newton method is also given, which is globally convergent. It is a generalization of the results in [3], [4, 13.3].

1. Introduction

Suppose we have a nonlinear system

$$f(x)=0, \tag{1}$$

where $f:D\subset R^n\to R^n$ is continuous on D. Moore and L. Qi introduced interval tests for existence and uniqueness of a solution to a nonlinear system in [1, 2]. However, the interval arithmetic is complicated. In this paper, some n-dimensional order interval iterative methods are presented. They can also be used as interval tests for existence and uniqueness of the solution to (1). Since they use endpoint calculation instead of interval arithmetic, they are simple.

In section 2 a simple interval Newton method and its global convergence is given. In section 3, an order interval Newton method is presented, which is a generalization of the Newton monotone iterative method given in [3], [4, 13.3].

The notation is as follows. Let R^n be the n-dimensional real space and $L(R^n)$ the space of all real $n\times n$ matrices. For vectors $x, y\in R^n$ and matrices $A, B\in L(R^n)$, we denote the usual componentwise partial orderings by $x\leqslant y$ and $A\leqslant B$. If $AB\leqslant I$ ($BA\leqslant I$), where I is the identity matrix, then A is called a left (right) subinverse of B. If A is both a left and a right subinverse of B, then A is called a subinverse of B.

Let $X=[\underline{x}, \bar{x}]=\{u|\underline{x}\leqslant u\leqslant \bar{x}\}$ be an n-dimensional interval vector; it is an order interval. $W(X)=\bar{x}-\underline{x}$ is called the width of the interval vector $X=[\underline{x}, \bar{x}]$, which is a nonnegative vector. We have the following properties of $W(\cdot)$:

(1) $W(\lambda X)=\lambda W(X)$, $\lambda\in R^1$ and $\lambda\geqslant 0$;

(2) $W(x+X)=W(X)$, $x\in R^n$;

(3) $W\left(\sum_{j=1}^{m} X^j\right)=\sum_{j=1}^{m} W(X^j)$;

(4) If $X\subset Y$, then $W(X)\leqslant W(Y)$.

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2. Order Interval Test and Simple Interval Newton Method

Let $X=[\underline{x}, \bar{x}]$ be an arbitrary order interval, i. e. $X:=\{u \mid \underline{x}\leqslant u\leqslant \bar{x}\}$. $P\in L(R^n)$ is a nonsingular matrix.

Define

$$Nu:=u-Pf(u),\ \forall u\in R^n \tag{2}$$

and

$$NX:=[N\underline{x},\ N\bar{x}] \tag{3}$$

for any order interval $X=[\underline{x}, \bar{x}]$. Then N is an interval operator.

Lemma 2.1. *Suppose $f:D\subset R^n\to R^n$ is continuous and there is a matrix $A\in L(R^n)$ such that*

$$f(\bar{x})-f(\underline{x})\leqslant A(\bar{x}-\underline{x}),\quad \underline{x}\leqslant\bar{x},\quad \underline{x},\ \bar{x}\in D. \tag{4}$$

If A has nonnegative, nonsingular, left subinverse P, then

$$\{Nu \mid u\in X\}\subset NX \tag{5}$$

for any $X=[\underline{x}, \bar{x}]\subset D$.

Proof. $\forall u\in X=[\underline{x}, \bar{x}]\subset D$, by (4), we have

$$N\bar{x}-Nu=\bar{x}-u-P(f(\bar{x})-f(u))\geqslant\bar{x}-u-PA(\bar{x}-u)\geqslant 0,$$

$$Nu-N\underline{x}=u-\underline{x}-P(f(u)-f(\underline{x}))\geqslant u-\underline{x}-PA(u-\underline{x})\geqslant 0,$$

i. e. $N\underline{x}\leqslant Nu\leqslant N\bar{x}$, i. e. (5) holds.

Lemma 2.2. *Suppose the conditions of Lemma 2.1 hold and $X=[\underline{x}, \bar{x}]\subset D$ is an order interval. Then NX contains all solutions of (1) in X. If $NX\subset X$, then there is a solution of (1) in X. If $X\cap NX=\emptyset$, then there is no solution of (1) in X.*

Proof. Suppose x^* is a solution of (1), $x^*\in X$. Then

$$x^*=x^*-Pf(x^*)=Nx^*\in NX.$$

Therefore, NX contains all solutions of (1) in X. This implies the last conclusion directly. Since f is continuous, so is $Nx=x-Pf(x)$. By (4) and Brouwer's fixed point theorem, we know that N has a fixed point in NX if $NX\subset X$. But all the fixed points of N are solutions of (1) and vice versa. This proves the second conclusion.

Lemma 2.3. *Suppose the conditions of Lemma 2.1 hold and $NX\subset X$; then*

$$N(NX)\subset NX. \tag{6}$$

Proof. By (5), we have

$$N(NX)=[N(N\underline{x}),\ N(N\bar{x})]\subset NX$$

since $N\underline{x},\ N\bar{x}\in NX\subset X$.

Now we construct simple interval Newton algorithm:

Algorithm 2.1. Let $X^0=[\underline{x}^0, \bar{x}^0]\subset D$. For $k=0, 1, \cdots$, if $X^k\cap NX^k=\emptyset$, then stop; otherwise, let $X^{k+1}=X^k\cap NX^k$.

Theorem 2.1. *Suppose the conditions of Lemma 2.1 hold and $X^0\subset D$ is an order interval, $\{X^k,\ k=0, 1, \cdots,\}$ is produced by Algorithm 2.1. Then all the solutions of (1) in X^0 are also in X^k for any nonnegative integer k. If $X^k\cap NX^k=\emptyset$ for a certain k, then there is no solution of (1) in X^0. If $NX^k\subset X^k$ for a certain k, then*