CORRECTION PROCEDURE FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS*

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The correction procedure has been discussed by L. Fox[6] and V. Pereyra[9] for accelerating the convergence of a certain approximate solution. Its theoretical basis is the existence of an asymptotic expansion for the error of discretization proved by Filippov and Rybinskii[10] and Stetter[11] (and Bohmer[2] for the general regions):

$$u-u_h=h^2v+O(h^4),$$

where u is the solution of the original differential equation, u_h the solution of the approximate finite difference equation with parameter h and v the solution of a correction differential equation independent of h. Stetter et al. used the extrapolation procedure to eliminate the auxiliary function v while Pereyra et al. used some special procedure to solve v approximately.

In the present paper we will present a difference procedure for solving v easily.

1. Difference Operator and Truncation Error

Let Δ_h be the 5-point approximation of the Laplace operator Δ in the 2-dimensional case and the 7-point approximation in the 3-dimensional case as usual[5].

Let Δ_h^* be the 5-point approximation defined by

$$\Delta_h^*u(x_1, x_2)=(\Sigma u(x_1\pm h, x_2\pm h)-4u(x_1, x_2))/2h^2,$$

$$\Sigma u(x_1\pm h, x_2\pm h)=u(x_1+h, x_2+h)+u(x_1-h, x_2-h)+u(x_1-h, x_2+h)+u(x_1+h, x_2-h)$$

in the 2-dimensional case and the 9-point approximation defined by

$$\Delta_h^*u(x_1, x_2, x_3)=(\Sigma u(x_1\pm h, x_2\pm h, x_3\pm h)-8u(x_1, x_2, x_3))/4h^2$$

in the 3-dimensional case.

Let δ_x, δ_y and δ_x^*, δ_y^* be the 2 and 4-point approximation defined respectively by

$$\delta_x u=(u(x+h, y)-u(x-h, y))/2h,$$

$$\delta_y u=(u(x, y+h)-u(x, y-h))/2h,$$

$$\delta_x^* u=(u(x+h, y-h)-u(x-h, y-h)+u(x+h, y+h)-u(x-h, y+h))/4h,$$

$$\delta_y^* u=(u(x-h, y-h)-u(x-h, y+h)+u(x+h, y-h)-u(x+h, y+h))/4h.$$

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The truncation error will be

$$(\Delta_h-\Delta)u=\frac{h^2}{12}E_1(u)+O(h^4), \tag{1}$$

$$(\Delta_h^*-\Delta)u=\frac{h^2}{12}E_2(u)+O(h^4), \tag{2}$$

$$\left(\delta_x-\frac{\partial}{\partial x}\right)u=\frac{h^2}{6}E_3(u)+O(h^4), \tag{3}$$

$$\left(\delta_y-\frac{\partial}{\partial y}\right)u=\frac{h^2}{6}E_4(u)+O(h^4), \tag{4}$$

$$\left(\delta_x^*-\frac{\partial}{\partial x}\right)u=\frac{h^2}{6}E_5(u)+O(h^4), \tag{5}$$

$$\left(\delta_y^*-\frac{\partial}{\partial y}\right)u=\frac{h^2}{6}E_6(u)+O(h^4), \tag{6}$$

with

$$E_1(u)=\sum\frac{\partial^4}{\partial x_i^4}u,\qquad E_2(u)=\sum\frac{\partial^4}{\partial x_i^4}u+6\sum_{i<j}\frac{\partial^2}{\partial x_i^2}\frac{\partial^2}{\partial x_j^2}u,$$

$$E_3(u)=\frac{\partial}{\partial x}\Delta u-\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}u,\qquad E_4(u)=\frac{\partial}{\partial y}\Delta u-\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2}u$$

$$E_5(u)=\frac{\partial}{\partial x}\Delta u+2\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}u,\qquad E_6(u)=\frac{\partial}{\partial y}\Delta u+2\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2}u.$$

we have

$$\frac{2}{3}E_1(u)+\frac{1}{3}E_2(u)=\Delta^2u, \tag{7}$$

$$\frac{2}{3}E_3(u)+\frac{1}{3}E_5(u)=\frac{\partial}{\partial x}\Delta u, \tag{8}$$

$$\frac{2}{3}E_4(u)+\frac{1}{3}E_6(u)=\frac{\partial}{\partial y}\Delta u. \tag{9}$$

2. Second Order Elliptic Problem

Consider

$$\Delta u=f(x,u)\ \text{in}\ \Omega,\qquad u=g\quad\text{on}\ \partial\Omega \tag{10}$$

in the 1, 2 or 3-dimensional domain Ω with boundary $\partial\Omega$. Suppose that Ω consists of in the 1, 2 or 3-dimensional domain Ω with boundary $\partial\Omega$. Suppose that Ω consists of some squares in the 2-dimensional case and of some cubes in the 3-dimensional case, and that the solution u is smooth enough and

$$f_1(u)=f_u'(x,u)\geq 0.$$

Consider the finite difference solution u_h defined by

$$\Delta_h u_h=f(x,u_h)\quad\text{in}\ \Omega_h,\qquad u_h=g-\frac{h^2}{12}f(x,g)\quad\text{on}\ \partial\Omega_h \tag{11}$$

and a correction solution φ_h defined by the linearized finite difference equation

$$(\Delta_h^*-f_1(u_h))\varphi_h=f(x,u_h)-f_1(u_h)u_h+\frac{h^2}{4}f_1(u_h)f(x,u_h)\quad\text{in}\ \Omega_h,\qquad \varphi_h=g-\frac{h^2}{12}f(x,g)\quad\text{on}\ \partial\Omega_h \tag{12}$$

with lattice domain Ω_h and the boundary $\partial\Omega_h$ of Ω_h defined as usual[10].