Note

AN ELEMENTARY PROOF OF THE CONVERGENCE FOR THE GENERALIZED BERNSTEIN-BÉZIER POLYNOMIALS*

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In [3], the author defines

$$B_{n,\alpha}(g;x)=g(0)+\sum_{k=1}^{n}\left[g\left(\frac{k}{n}\right)-g\left(\frac{k-1}{n}\right)\right]f_{n,k}^{\alpha}(x) \tag{1}$$

for $\alpha>0$ and calls (1) the generalized Bernstein-Bézier polynomial of the function $g(x)$ defined in [0, 1]. Polynomials involved in (1)

$$f_{n,k}(x)=\sum_{i=k}^{n}J_{n,i}(x),\quad k=0, 1, \cdots, n \tag{2}$$

are called n-th Bézier basis functions[1-3], where

$$J_{n,i}(x)=\binom{n}{i}x^{i}(1-x)^{n-i},\quad i=0, 1, \cdots, n. \tag{3}$$

When $\alpha=1$, (1) reduces to the classical Bernstein polynomial for $g(x)$. It is proven in [3] that the following theorem is valid.

Convergence Theorem. *For any $\alpha>0$ and $g\in C[0, 1]$, we have*

$$\lim_{n\to\infty}B_{n,\alpha}(g;x)=g(x)\quad \textit{uniformly in}\quad [0, 1]. \tag{4}$$

The outline of the proof for the theorem is as follows. We point out there that $B_{n,\alpha}$ is a positive linear operator and then show that

$$\lim B_{n,\alpha}(x^{i};x)=x^{i},\quad i=0, 1, 2,$$

uniformly in [0, 1]. The convergence theorem follows as soon as the Korovkin theorem is applied. It is clear that $B_{n,\alpha}(1;x)=1$. As we can see in [3], the crucial step is to verify $\lim B_{n,\alpha}(x;x)=x$ uniformly in [0, 1], a fact which is equivalent to the following

Main Theorem. *For $\alpha>0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f_{n,k}^{\alpha}(x)=x\quad \textit{uniformly in } [0, 1]. \tag{5}$$

In [3], (5) is proven by using the Tchebychev inequality so called in the probability theory. A purely analytical proof presented in this paper, in which some new results of the Bézier basis functions are involved, seems to be desirable and to have independent interest.

Two known identities of the Bézier basis functions

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$$\frac{1}{n}\sum_{k=1}^{n} f_{n,k}(x)=x \tag{6}$$

and

$$f_{n,k}(x)+f_{n,n+1-k}(1-x)=1, \quad k=0,1,\cdots,n \tag{7}$$

will be useful in the sequel. The proofs for (6) and (7) could be found in [2] and [1] respectively. Consider the case of $\alpha=2$ at first. By (6) and (7) we have

$$0\leqslant x-\frac{1}{n}\sum_{k=1}^{n} f_{n,k}^2(x)=\frac{1}{n}\sum_{k=1}^{n} f_{n,k}(x)[1-f_{n,k}(x)]$$

$$=\frac{1}{n}\sum_{k=1}^{n} f_{n,k}(x)f_{n,n+1-k}(1-x). \tag{8}$$

Setting
$$h_n(x)=\sum_{k=1}^{n} f_{n,k}(x)f_{n,n+1-k}(1-x),$$

this function can also be formulated by

$$h_n(x)=[f_{n,n}(1-x),\ \cdots,\ f_{n,1}(1-x)]F_n(x), \tag{9}$$

where
$$F_n(x)=[f_{n,1}(x),\ f_{n,2}(x),\ \cdots,\ f_{n,n}(x)]^t.$$

Put $l_n=[1,1,\cdots,1]^t$ and

$$B(x)=xI+(1-x)E, \tag{10}$$

where I denotes the $n\times n$ identity matrix and

$$E=\begin{bmatrix} 0 & 1 & 0\cdots 0\\ 0 & 0 & 1\cdots 0\\ \cdots & \cdots & \cdots\\ 0 & 0 & 0\cdots 1\\ 0 & 0 & 0\cdots 0\end{bmatrix}.$$

In [2] we have shown that

$$[B(x)]^k l_n=\begin{bmatrix} l_{n-k}\\ F_k(x)\end{bmatrix}, \quad k\leqslant n, \tag{11}$$

similarly

$$l_n^t[B(1-x)]^k=[f_{k,k}(1-x),\ \cdots,\ f_{k,1}(1-x),\ l_{n-k}^t] \tag{12}$$

for $k\leqslant n$. Since $B(x)$ and $B(1-x)$ both are polynomials of E, they are commutative in multiplication and we have from (9) by (11) and (12)

$$h_n(x)=l_n^t[B(1-x)B(x)]^n l_n. \tag{14}$$

Since
$$\frac{d}{dx}[B(1-x)B(x)]^n=n(1-2x)[B(1-x)B(x)]^{n-1}(I-E)^2,$$

we have by (11), (12) and (2) that

$$h_n'(x)=n(1-2x)\sum_{k=1}^{n} J_{n-1,n-k}(1-x)J_{n-1,k-1}(x). \tag{15}$$

We conclude from (15) that $h_n(x)$ has maximum at $x=1/2$, thus

$$0\leqslant h_n(x)\leqslant h_n(1/2) \quad \text{for } 0\leqslant x\leqslant 1. \tag{16}$$

From (14) we get

$$h_n(1/2)=l_n^t[B(1/2)]^{2n}l_n=(1/4)^n l_n^t(I+E)^{2n}l_n=(1/4)^n l_n^t\sum_{k=0}^{2n}\binom{2n}{k}E^k l_n.$$

Since $E^k=0$ for $k\geqslant n$, we have