A CLASS OF ITERATION METHODS FOR A STRONGLY MONOTONE OPERATOR EQUATION AND APPLICATION TO FINITE ELEMENT APPROXIMATE SOLUTION OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM*

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1. Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and reduced norm $\|\cdot\|$. An operator $A: H \rightarrow H$ is said to be strongly monotone with modulus $\alpha > 0$ if

$$(Ax-Ay,\ x-y) \geqslant \alpha \|x-y\|^2, \qquad \forall x,\ y \in H.$$

It is said to be monotone if the inequality above is valid for $\alpha = 0$. Furthermore, it is maximal monotone if it is monotone and its monotone proper extension does not exist.

Let A be a strongly monotone operator as above, and $b \in H$ a definite element. The present article is devoted to a study of the operator equation

$$Ax = b, \tag{1.1}$$

which is often deduced from practical problems in the field of differential equation, variational method and optimal control (cf. [5], [6] and [7]). The iteration schemes we will use is

$$x_{n+1} = x_n + t_n(b - Ax_n), \tag{1.2}$$

where $\{t_n\}$ is a parameter sequence of positive reals.

A special form of equation (1.1), $x + Bx = b$, where B is a monotone operator, has been discussed recently by several authors. Using the schemes (1.2), R. E. Bruck, Jr.[1] proved its local convergence on the assumption that the equation is solvable; W. G. Dotson, Jr.[2] assumed B to be nonexpansive; and You Zhao-yong[3] relaxed B into being Lipschitz continuous and then proved its global convergence. Instead of (1.2), O. Nevanlinna[4] applied the iteration

$$x_{n+1} = x_n + t_n(b - Ax_n + \theta_n x_n) \tag{1.3}$$

and, supposing that B is continuous and bounded and satisfies a linear growth condition respectively, proved the global convergence of (1.3). In his results, for a continuous B, it is required that at each iteration step the parameters t_n and θ_n be

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chosen on an optimum condition which is concerned with the operator itself, and, for a bounded B, use of a so called reinitalization processes demanded.

However, in practice, the local convergence and the dependence of the choice of the parameter t_n upon the operator itself constantly cause inconvenience, and in many practical problems such as the nonlinear elliptic boundary value problem to be studied in section 3, neither the Lipschitz continuity nor the boundedness, nor even the general continuity, can be guaranteed for the operator. Therefore, we will try here to weaken the assumption of continuity, boundedness and linear growth condition on A (for equation (1.1)), and on this basis establish the global convergence of (1.2) with the choice of the parameter being independent of the operator. As an application, we will also try to provide a convenient and efficient iteration method for finite element approximate solution of the elliptic boundary value problem (1.3).

2. Global Convergence Theorems

We first introduce the following definitions:

Definition 1. *Let $g: H \to [0, \infty)$ be a functional which maps any bounded closed convex set in H into a bounded set in $[0, \infty)$. An operator $A: H \to H$ is said to be upper controlled by the functional g if*

$$\|Ax\| \leqslant g(x)$$

is valid for every $x \in H$.

Definition 2. *Let $\varphi: [0, \infty) \to [0, \infty)$ be a continuous real function with the property $\varphi(0)=0$. An operator $A: H \to H$ is said to have the continuity of the upper controlled function φ provided*

$$\|Ax - Ay\| \leqslant \varphi(\|x-y\|)$$

is valid for every $x, y \in H$.

From the definitions above, an operator having the continuity of an upper controlled function φ is also upper controlled by the functional $g(x)=\varphi(\|x-y^0\|+\|Ay^0\|)$ where y^0 is arbitrary in H. For an operator A that satisfies respectively the boundedness condition (i.e., A maps bounded sets into bounded sets), and linear growth condition $\|Ax\| \leqslant c(1+\|x\|)$ (one of the assumptions of [4]), there naturally exist functionals $g(x)=\|Ax\|$ and $g(x)=c(1+\|x\|)$ such that A is upper controlled by g. Also, an operator satisfying the Lipschitz condition $\|Ax-Ay\| \leqslant L\|x-y\|$ (the assumption in [3]) has the continuity of the upper controlled function $\varphi(t)=Lt$, and an operator which is upper controlled by a functional g may be discontinuous.

Set

$$R(x) = 1/\alpha(\|b - Ax\|), \tag{2.1}$$

$$U(x, \delta) = \{y \in H \mid \|y\| \leqslant (R(x)+\delta)^{1/2}\}, \tag{2.2}$$

$$M(x) = \sup\{g(y) \mid \|y\| \leqslant (R(x)+\delta)^{1/2} + R(0)\}, \tag{2.3}$$

where $x \in H$ is arbitrary and δ is a positive real number.

Theorem 1. *Suppose that the strongly monotone operator A is upper controlled by a functional g and equation (1.1) is solvable. For any initial value $x_0 \in H$, choose a sequence of positive reals $\{\bar{t}_n\}$ satisfying*