THE ESTIMATES OF $\|M^{-1}N\|_\infty$ AND THE OPTIMALLY SCALED MATRIX*

HU JIA-GAN (胡家赣)

(*Institute of Computational Mathematics and Applied Physics, Beijing, China*)

§ 1. Introduction

It is well-known that, if the elements m_{ij} of an $n\times n$ matrix M satisfy the inequality

$$|m_{ii}|-\sum_{j\neq i}|m_{ij}|\geqslant\delta>0, \qquad i=1, 2, \cdots, n, \tag{1}$$

where δ is a constant, then the inequality

$$\|M^{-1}\|_\infty\leqslant 1/\delta \tag{2}$$

holds[1,2]. But sometimes it is necessary to estimate the norm $\|M^{-1}N\|_\infty$, where N is an $n\times n$ or $n\times m$ matrix, and the use of the estimate (2), i.e. the estimate $\|M^{-1}N\|_\infty\leqslant\|M^{-1}\|_\infty\|N\|_\infty\leqslant\|N\|_\infty/\delta$, does not result satisfactorily. In this paper, we give an upper and a lower bound of $\|M^{-1}N\|_\infty$ for some matrices M and N. In [3], James and Riha applied the scaling transformation to prove the convergence of some iterative schemes for solving systems of linear algebraic equations. In this paper, we define an "optimally scaled matrix" by means of the scaling transformation. Our estimates of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix are very useful in the discussion of the convergence of some iterative matrices. For, in the literature up to now, in order to prove the convergence of an iterative matrix $G(A)$ of a matrix A, such as Jacobi iterative matrix, SOR iterative matrix, etc., it is a common procedure to construct a dominant matrix $H(A)$, such that $|G(A)|\leqslant H(A)$ and, consequently,

$$\rho(G(A))\leqslant\rho(H(A)), \tag{3}$$

where $\rho(\cdot)$ is the spectral radius of the matrix enclosed in the brackets; thus, we need only to prove the convergence of the iterative matrix $H(A)$. Now for the optimally scaled matrix $\widetilde{A}$ of the matrix A we have

$$\rho(G(A))=\rho(G(\widetilde{A})). \tag{4}$$

Evidently, (4) is better than (3), since from (4) $G(A)$ is convergent, if and only if $G(\widetilde{A})$ is so, and this may be obtained easily by our estimates of $\|M^{-1}N\|_\infty$. We will discuss in this way the convergence of some splittings of a matrix. Besides, we will give some other applications of the estimates of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix.

§ 2. The Estimates of $\|M^{-1}N\|_\infty$ and the Optimally Scaled Matrix

Theorem 1. *If $M=(m_{ij})$ is an $n\times n$ matrix, $N=(n_{ij})$ is an $n\times m$ matrix and*

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$$|m_{ii}|>\sum_{j\neq i}|m_{ij}|,\qquad i=1,\ 2,\ \cdots,\ n,\tag{5}$$

then we have

$$\|M^{-1}N\|_\infty\leqslant\max_i\Big(\sum_j|n_{ij}|/\big(|m_{ii}|-\sum_{j\neq i}|m_{ij}|\big)\Big),\tag{6}$$

where $$\|M^{-1}N\|_\infty=\max_i\sum_j|(M^{-1}N)_{ij}|.$$

Proof. First let $N=(n_i)$ be an $n\times 1$ matrix, that is, an n-dimensional vector, and $M^{-1}N=X$. Thus, $MX=N$. If $X=(x_1,\ x_2,\ \cdots,\ x_n)^T$ and

$$\|M^{-1}N\|_\infty=\|X\|_\infty=\max_i|x_i|=|x_{i_0}|,$$

then $$m_{i_0i_0}x_{i_0}=n_{i_0}-\sum_{j\neq i_0}m_{i_0j}x_j.$$

Hence, $$|m_{i_0i_0}|\,|x_{i_0}|\leqslant|n_{i_0}|+|x_{i_0}|\sum_{j\neq i_0}|m_{i_0j}|$$

and $$|x_{i_0}|\leqslant|n_{i_0}|/\big(|m_{i_0i_0}|-\sum_{j\neq i_0}|m_{i_0j}|\big)\leqslant\max_i\big(|n_i|/\big(|m_{ii}|-\sum_{j\neq i}|m_{ij}|\big)\big).$$

Thus we have proved (6) when N is an n-dimensional vector. Now, let $N=(n_{ij})$ be an $n\times m$ matrix, $|N|=(|n_{ij}|)$, $D=\operatorname{diag}M$, $B=D-M$, $\widetilde{M}=|D|-|B|$ and $\widetilde{N}=|N|$. It is easily seen that

$$\rho(D^{-1}B)\leqslant\rho(|D|^{-1}|B|)<1.$$

Therefore $$M^{-1}=(D-B)^{-1}=(I+D^{-1}B+(D^{-1}B)^2+\cdots)D^{-1},$$

$$\widetilde{M}^{-1}=(|D|-|B|)^{-1}=(I+|D|^{-1}|B|+(|D|^{-1}|B|)^2+\cdots)|D|^{-1}.$$

Hence $$|M^{-1}|\leqslant\widetilde{M}^{-1},\qquad|M^{-1}N|\leqslant\widetilde{M}^{-1}\widetilde{N}$$

and $$\|M^{-1}N\|_\infty\leqslant\|\widetilde{M}^{-1}\widetilde{N}\|_\infty$$

(we also have $\rho(M^{-1}N)\leqslant\rho(\widetilde{M}^{-1}\widetilde{N})$, provided that N is a square matrix). Now, if $\widetilde{N}_j$ is the vector composed of the elements of the jth column of $\widetilde{N}$, from $\widetilde{M}^{-1}\geqslant0$ we have

$$\|\widetilde{M}^{-1}\widetilde{N}\|_\infty=\max_i\sum_j(\widetilde{M}^{-1}\widetilde{N})_{ij}=\max_i\sum_j(\widetilde{M}^{-1}\widetilde{N}_j)=\max_i\big(\widetilde{M}^{-1}\sum_j\widetilde{N}_j\big).$$

Taking $\sum_j\widetilde{N}_j$ as the n-dimensional vector N and $\widetilde{M}$ as M mentioned above, we have proved our theorem.

In Theorem 1 taking $N=I$ (the unit matrix), we get the estimate (2). Thus (2) is a special case of (6).

Theorem 2. *Under the conditions of Theorem 1, if, furthermore, M is an L-matrix and $N\geqslant0$ then*

$$\min_i\sum_j(M^{-1}N)_{ij}\geqslant\min_i\Big(\sum_j|n_{ij}|/\big(|m_{ii}|-\sum_{j\neq i}|m_{ij}|\big)\Big)=\min_i\Big(\sum_j n_{ij}/\sum_j m_{ij}\Big).\tag{7}$$

The proof of this theorem is similar to that of Theorem 1 and is therefore omitted, but it may be noticed that, under the conditions of Theorem 2, $M=\widetilde{M}$, $N=\widetilde{N}$, $M^{-1}\geqslant0$ and $M^{-1}N\geqslant0$.

It is well-known that[4], if $A=(a_{ij})\geqslant0$, then

$$\min_i\sum_j a_{ij}\leqslant\rho(A)\leqslant\max_i\sum_j a_{ij}.\tag{8}$$

Therefore, we have

Corollary 1. If $M=(m_{ij})$ is an $n\times n$ L-matrix, $N=(n_{ij})$ is an $n\times n$ nonnegative matrix and $\sum_j m_{ij}>0$ $(i=1,\ 2,\ \cdots,\ n)$, then