THE COST OF KUHN'S ALGORITHM AND COMPLEXITY THEORY*

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Abstract

A comparison by Wang and Xu[6] between S. Smale's cost estimation for Newton's method and that of the author's for Kuhn's algorithm, both aiming at the zero finding of complex polynomials, showed improments the advantage of the latter in finding zeros and approximate zeros. In this paper, important on the above work are made. Furthermore, a probabilistic estimation of the monotonicity of Kuhn's algorithm is obtained.

§ 1. Introduction

A comparison between S. Smale's cost estimation for Newton's method and that of the author's for Kuhn's algorithm, both aiming at the zero finding of complex polynomials, was presented by Wang and Xu[6]. It turns out that the latter is much better both in finding zeros and in finding approximate zeros. The ratios are respectively from n^9/μ^7 to $n^2\log(n/\varepsilon)$ and from n^9/μ^7 to $n^3\log(n/\mu)$, where n is the degree of polynomials, $\varepsilon>0$ is the accuracy demand for resulted zeros, and μ is the probability allowing the corresponding estimation to fail, $0<\mu<1$.

James Renegar obtained similar results[3]. His Lemma 3.1 and Proposition 5.6 in [3].

Here we improve the results of Wang and Xu[6]. A new cost estimation for Kuhn's algorithm is given by Theorem 2.12 while Theorem 3.8 answers probabilistically the problem of finding the approximate zeros of polynomials suggested by S. Smale. This is followed by a discussion in Theorem 4.15 on a probabilistic estimation of the monotonicity of Kuhn's algorithm.

§ 2. Cost of Kuhn's Algorithm

In order to be consistent, we use the same notation as used by Kuhn[2]. For simplicity, let $\bar{z}=0$ and $h=1$.

The algorithm can be sketched as follows. The half-space $C\times[-1,\infty)$ is simplicially triangulated such that every vertex is in some plane $C_d=C\times\{d\}$, $d=-1, 0, 2, \cdots$, and each plane C_d is then subdivided into isosceles right triangles with right-angle sides equal to $s(d)$, where $s(-1)=1$ and $s(d)=2^{-d}$, $d\geqslant0$.

The labelling for vertices in C_d, $d\geqslant0$, is that (the argument of a complex

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number is restricted to $(-\pi, \pi]$)

$$l(z)=\begin{cases}1, & \text{if } f(z)=0 \text{ or } -\pi/3\leqslant\arg f(z)\leqslant\pi/3,\\ 2, & \text{if } \pi/3<\arg f(z)\leqslant\pi,\\ 3, & \text{if } -\pi<\arg f(z)<-\pi/3,\end{cases}$$

where $f(z)=z^n+\sum_{i=0}^{n-1}a_iz^i, a_i\in C$, while the labelling for C_{-1} uses z^n instead of $f(z)$.

Let Q be the square in C_{-1} bounded by $x=\pm m$ and $y=\pm m$, where $z=x+iy$ and $m=\lceil 3(1+\sqrt{2})n/4\pi\rceil$. The symbol $\lceil a\rceil$ is the least integer not less than a.

A triangle is said to be completely labelled (c.l. triangle) if its three vertices are exactly labelled 1, 2 and 3.

Proposition 2.1. Let Σ_d be an elementary cube of the triangulation between C_d and C_{d+1}, and let B_d be a cylinder with axis $\{0\}\times[d, d+1]$. Let σ_d denote the number of tetrahedra which belong to Σ_d and are wholly contained in B_d. Then

$$\sigma_d\leqslant\begin{cases}5\cdot\mathrm{vol}(\Sigma_d\cap B_d), & \text{if } d=-1,\\ 14\cdot\mathrm{vol}(\Sigma_d\cap B_d)\cdot 2^{2d}, & \text{if } d\geqslant 0.\end{cases}$$

Proof. Let $V_d=\mathrm{vol}(\Sigma_d\cap B_d)$ for convenience. For $d=-1$, we always have $1\geqslant V_d\geqslant 0$. Obviously, if $V_d\leqslant 1/2$, then $\sigma_d=0$; if $V_d=1$, then $\Sigma_d\subset B_d$. So $\sigma_d=5$. In the case $1/2<V_d<1$, we have $\sigma_d\leqslant 1$ because every vertical edge of Σ_d touches four tetrahedra contained in Σ_d. In any of the above cases, the proposition is true.

For $d\geqslant 0$, $0\leqslant V_d\leqslant 2^{-2d}$. We discuss eight cases. Since the central point of the upper square is touched by all the fourteen tetrahedra of the cube, it is easy to obtain the corresponding results for σ_d by simple volume analysis:

(1) $V_d=2^{-2d}$, $\quad\sigma_d=14$;

(2) $\frac{7}{8}2^{-2d}<V_d<2^{-2d}$, $\quad\sigma_d\leqslant 9$;

(3) $\frac{3}{4}2^{-2d}<V_d\leqslant\frac{7}{8}2^{-2d}$, $\quad\sigma_d\leqslant 8$;

(4) $\frac{1}{2}2^{-2d}<V_d\leqslant\frac{3}{4}2^{-2d}$, $\quad\sigma_d\leqslant 7$;

(5) $\frac{3}{8}2^{-2d}<V_d\leqslant\frac{1}{2}2^{-2d}$, $\quad\sigma_d\leqslant 4$;

(6) $\frac{1}{4}2^{-2d}<V_d\leqslant\frac{3}{8}2^{-2d}$, $\quad\sigma_d\leqslant 3$;

(7) $\frac{1}{8}2^{-2d}<V_d\leqslant\frac{1}{4}2^{-2d}$, $\quad\sigma_d\leqslant 1$;

(8) $0\leqslant V_d\leqslant\frac{1}{8}2^{-2d}$, $\quad\sigma_d=0$.

In all the eight cases, $\sigma_d\leqslant 14\cdot V_d\cdot 2^{2d}$. It completes the proof.

Lemma 2.2. *Suppose that w_k are complex constants, $k=1, 2, 3$, and $-\pi/3\leqslant\arg w_1\leqslant\pi/3$; $\pi/3<\arg w_2\leqslant\pi$; $-\pi<\arg w_3<-\pi/3$. Then $\left|\arg\frac{w_2}{w_1}\right|\geqslant\frac{2\pi}{3}$, or $\left|\arg\frac{w_2}{w_3}\right|\geqslant\frac{2\pi}{3}$, or $\left|\arg\frac{w_1}{w_3}\right|\geqslant\frac{2\pi}{3}$.*

Proof. Since

$$\left|\arg\frac{w_2}{w_1}\right|<\frac{2\pi}{3}\Leftrightarrow 0<\arg w_2-\arg w_1<\frac{2\pi}{3}, \tag{2.1}$$