CANONICAL BOUNDARY ELEMENT METHOD FOR PLANE ELASTICITY PROBLEMS*

YU DE-HAO (余德浩)

(Computing Center, Academia Sinica, Beijing, China)

Abstract

In this paper, we apply the canonical boundary reduction, suggested by Feng Kang[1], to the plane elasticity problems, find the expressions of canonical integral equations and Poisson integral formulas in some typical domains. We also give the numerical method for solving these equations together with their convergence and error estimates. Coupling with classical finite element method, this method can be applied to other domains.

Introduction

The plane elasticity problems include the plane stress problem and plane strain problem. They have an unified mathematical formulation[2].

Taking displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in directions x_1 and x_2 as basic unknown functions, we can give the expressions of strain ε_{ij}, $i, j=1, 2$, and stress σ_{ij}, $i, j=1, 2$. Consider the equilibrium equations with traction boundary condition

$$\begin{cases} (\lambda+2\mu)\ \text{grad div}\ \boldsymbol{u}-\mu\ \text{rot rot}\ \boldsymbol{u}=0, & \text{in } \Omega, \\ \sum_{j=1}^{2}\sigma_{ij}n_j=g_i, \quad i=1,2, & \text{on } \Gamma, \end{cases} \tag{1}$$

where Ω is a domain with smooth boundary Γ, λ and μ are Lamè coefficients, (n_1, n_2) are the outward normal direction cosines on Γ. Let

$$V(\Omega)=H^1(\Omega)^2,$$

$$\mathscr{R}=\left\{\boldsymbol{v}\in V(\Omega)\,\middle|\,\boldsymbol{v}=\begin{bmatrix} c_1-c_3x_2 \\ c_2+c_3x_1\end{bmatrix},\ c_1,\ c_2,\ c_3\in R\right\},$$

where $H^1(\Omega)$ is the Sobolev space, then the boundary value problem (1) is equivalent to the variational problem

$$\begin{cases} \text{Find } \boldsymbol{u}\in V(\Omega) \text{ such that} \\ D(\boldsymbol{u}, \boldsymbol{v})=F(\boldsymbol{v}), \quad \forall \boldsymbol{v}\in V(\Omega), \end{cases} \tag{2}$$

where

$$D(\boldsymbol{u}, \boldsymbol{v})=\iint_\Omega \sum_{i,j=1}^{2}\sigma_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\boldsymbol{v})\,dp$$

$$=\iint_\Omega \left\{2\mu\sum_{i,j=1}^{2}\varepsilon_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\boldsymbol{v})+\lambda\sum_{k=1}^{2}\varepsilon_{kk}(\boldsymbol{u})\,\varepsilon_{kk}(\boldsymbol{v})\right\}dp,$$

$$F(\boldsymbol{v})=\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{v}\,ds.$$

* Received December 1, 1983.

$D(\boldsymbol{u}, \boldsymbol{v})$ is a semi-positive definite symmetric bilinear form, and $D(\boldsymbol{v}, \boldsymbol{v})=0$ if and only if $\varepsilon_{ij}(\boldsymbol{v})=0$, $i, j=1, 2$, and if and only if $\boldsymbol{v}\in\mathscr{R}$.

The sufficient and necessary condition for having solution of the variational problem (2) is

$$F(\boldsymbol{v})=0, \quad \forall \boldsymbol{v}\in\mathscr{R}.$$

Thus we obtain the consistency condition of boundary traction as follows

$$\begin{cases} \int_\Gamma g_i\, ds=0, & i=1, 2, \\ \int_\Gamma (x_1 g_2 - x_2 g_1)\, ds=0. \end{cases} \tag{3}$$

From now on we always assume that (3) is satisfied.

Let

$$V^{\cdot}(\Omega)=V(\Omega)/\mathscr{R},$$
$$F^{\cdot}(\boldsymbol{v}^{\cdot})=F(\boldsymbol{v}), \quad \boldsymbol{v}\in\boldsymbol{v}^{\cdot},$$
$$D^{\cdot}(\boldsymbol{u}^{\cdot}, \boldsymbol{v}^{\cdot})=D(\boldsymbol{u}, \boldsymbol{v}), \quad \boldsymbol{u}\in\boldsymbol{u}^{\cdot}, \quad \boldsymbol{v}\in\boldsymbol{v}^{\cdot},$$

we consider the variational problem (2) in the quotient space $V^{\cdot}(\Omega)$, i.e.

$$\begin{cases} \text{Find } \boldsymbol{u}^{\cdot}\in V^{\cdot}(\Omega) \text{ such that} \\ D^{\cdot}(\boldsymbol{u}^{\cdot}, \boldsymbol{v}^{\cdot})=F^{\cdot}(\boldsymbol{v}^{\cdot}), \quad \forall \boldsymbol{v}^{\cdot}\in V^{\cdot}(\Omega). \end{cases} \tag{4}$$

Using the Korn's inequality[4], we have

Proposition 1. $D^{\cdot}(\boldsymbol{v}^{\cdot}, \boldsymbol{v}^{\cdot})\geqslant\alpha\|\boldsymbol{v}^{\cdot}\|^2_{V^{\cdot}}, \forall \boldsymbol{v}^{\cdot}\in V^{\cdot}(\Omega)$,
where α is a positive constant.

Then by Lax-Milgram Lemma we obtain immediately

Proposition 2. The variational problem (4) has one and only one solution, and the solution depends on given traction continuously.

1. Canonical Boundary Reduction

Take the plane Cartesian coordinates. Define a differential operator L and a differential boundary operator β as follows:

$$L=\begin{bmatrix} a\dfrac{\partial^2}{\partial x^2}+b\dfrac{\partial^2}{\partial y^2} & (a-b)\dfrac{\partial^2}{\partial x\partial y} \\ (a-b)\dfrac{\partial^2}{\partial x\partial y} & b\dfrac{\partial^2}{\partial x^2}+a\dfrac{\partial^2}{\partial y^2} \end{bmatrix},$$
$$\beta=\begin{bmatrix} an_1\dfrac{\partial}{\partial x}+bn_2\dfrac{\partial}{\partial y} & (a-2b)n_1\dfrac{\partial}{\partial y}+bn_2\dfrac{\partial}{\partial x} \\ (a-2b)n_2\dfrac{\partial}{\partial x}+bn_1\dfrac{\partial}{\partial y} & an_2\dfrac{\partial}{\partial y}+bn_1\dfrac{\partial}{\partial x} \end{bmatrix}_\Gamma, \tag{5}$$

where $a=\lambda+2\mu$, $b=\mu$, then the boundary value problem (1) can be written as

$$\begin{cases} L\boldsymbol{u}=0, & \text{in } \Omega, \\ \beta\boldsymbol{u}=\boldsymbol{g}, & \text{on } \Gamma. \end{cases} \tag{6}$$

We have Green's formula

$$\iint_\Omega \boldsymbol{v}\cdot L\boldsymbol{u}\, dp=\int_\Gamma \boldsymbol{v}\cdot\beta\boldsymbol{u}\, ds-D(\boldsymbol{u}, \boldsymbol{v}) \tag{7}$$