SEMI-LINEAR DIFFERENCE SCHEMES*

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Abstract

A class of semi-linear numerical differentiation formulas is designed for functions with steep gradients. A semi-linear second-order difference scheme is constructed to solve the two-point singular perturbation problem $-\epsilon u''+p(x)u'+q(x)u=f(x)$, $u(0)=u(1)=0$. It is shown that this semi-linear scheme has one more order of approximation precision than the central difference scheme for small ϵ and saves computation time for required accuracy. Numerical results agreeing with the above analysis are included.

1. Introduction

Numerical differential formulas play a very important role in constructing difference schemes of differential equations. However, the usual numerical differentiation formulas based on polynomial approximations may lead to very poor results when the functions are not smooth. Usually there are two ways to avoid this difficulty; one is to refine the mesh, and the other is to use a higher order polynomial interpolation.

A different approach is introduced in this paper by considering "weak" nonlinear numerical differentiation formulas beyond linear functional approximation. In section 2, we derive some semi-linear numerical differentiation formulas. Such a scheme is semi-linear as an operator; besides, the numerical differentiation formula, chosen for a function with steep gradients, should depend on the behavior of the function. As an example, a detailed analysis is given in section 3 for a model problem: $-\epsilon u''+u'=0$, $u(0)=0$, $u(1)=1$, which was recently discussed by other authors[1]-[3]. In section 4, we consider a more general elliptic singular perturbation problem: $-\epsilon u''+pu'+qu=f$. The semi-linear scheme presented there is shown to have one more order of precision than the conventional central difference scheme for the singular perturbation problem if $h\leqslant 2\epsilon/\|p\|_\infty$, where h is a uniform mesh size in the "singular" subdomain. In the larger "regular" subdomain the mesh size can be used as large as one desires. While maintaining the same accuracy, the semi-linear scheme costs less CPU time than the linear scheme. There is a simple way to reduce the resulting semi-linear system to an iteration with the corresponding linear system. The numerical tests presented in section 5 match the above analysis very well. A similar study in the two-dimensional case will appear in another separate paper.

2. Semi-linear Numerical Differentiation Formulas

Let $u(x)$ be a function defined in (a, b) with large derivatives. Without loss of

* Received September 9, 1982.

1) This research was supported in part by ONR Grant N00014-82-K-0184, and Yale University when the author was visiting the Dept. of Computer Science of Yale.

generality, suppose $u(x)$ is monotonic in the interval, and $u=Fx$ is a one-to-one map.

Denote $x_{-1}=a<x_0<x_1=b$, and let G be defined as an indefinite integral of F such that

$$G(x)=\int Fx\,dx. \tag{1}$$

By the mean value theorem, there exist two points z_{-1} and z_1: $x_{-1}<z_{-1}<x_0<z_1<x_1$, such that

$$[F^{-1}u_0,\ F^{-1}u_{-1}]G=u(z_{-1}),\quad [F^{-1}u_1,\ F^{-1}u_0]G=u(z_1), \tag{2}$$

where $[x_1,\ x_2]Y$ denotes the divided difference: $[x_1,\ x_2]Y=\dfrac{Y(x_1)-Y(x_2)}{x_1-x_2}$. Now we look for an approximate formula for the first derivative at the node $x=x_0$, based on the formulas (2), as follows:

$$u'(x_0)\sim\frac{2}{x_1-x_{-1}}([F^{-1}u_1,\ F^{-1}u_0]G-[F^{-1}u_0,\ F^{-1}u_{-1}]G). \tag{3}$$

If we take F as the identity map, and $G(x)=x^2/2$, then (3) is just the usual central difference formula based on the quadratic interpolation. In general, we assume F to be an admissible one-to-one map such that G can be obtained from (1) directly. For any such F, (3) defines a numerical formula for the first derivative at the node $x=x_0$. As an example, let $F^{-1}f=f^r$, where r is a real parameter. Suppose $u(x)>0$ in $(x_{-1},\ x_1)$; from (3) and (1), we obtain

$$u'(x_0)\doteq\frac{2r}{(1+r)(x_1-x_{-1})}\left\{\frac{u_1^{1+r}-u_0^{1+r}}{u_1^r-u_0^r}-\frac{u_0^{1+r}-u_{-1}^{1+r}}{u_0^r-u_{-1}^r}\right\}, \tag{4}$$

where $u_j=u(x_j)$, $j=-1,\ 0,\ 1$.

When $r=\dfrac{1}{2}$, $x_0=\dfrac{1}{2}(x_1+x_{-1})$, (4) becomes

$$u'(x_0)=\frac{1}{3h}(u_1^{1/2}-u_{-1}^{1/2})(u_1^{1/2}+u_0^{1/2}+u_{-1}^{1/2}),\qquad h=x_1-x_0.$$

Theorem 1. *Let $u, F\in C^k(x_{-1},\ x_1)$ where $k=3$ or 4, $F^{-1}u$ is a one-to-one map, $h=x_0-x_{-1}=x_1-x_0$. Then, the remainder of the numerical differentiation formula* (3) *equals*

$$\frac{1}{h}([F^{-1}u_1,\ F^{-1}u_0]G-[F^{-1}u_0,\ F^{-1}u_{-1}]G)-u'(x_0)$$
$$=\frac{h^2}{12}\frac{d}{dx}\left\{2u''+u'^2\frac{d^2F^{-1}u}{du^2}\left(\frac{dF^{-1}u}{du}\right)^{-1}\right\}\Bigg|_{x=x_0}+O(h^k). \tag{5}$$

Proof. Applying the Taylor expansion for $G(y)$ upon z, one obtains

$$G(y)-G(z)=(y-z)G'(z)+(y-z)^2G''(z)/2+(y-z)^3G^{(3)}(z)/3!$$
$$+(y-z)^4G^{(4)}(z)/4!+O((y-z)^5).$$

Hence

$$W(y_1,\ y_0,\ y_{-1})\equiv\frac{G(y_1)-G(y_0)}{y_1-y_0}-\frac{G(y_0)-G(y_{-1})}{y_0-y_{-1}}$$
$$=\frac{y_1-y_{-1}}{2}G''(y_0)+\frac{(y_1-y_0)^3+(y_0-y_{-1})^3}{4!}G^{(4)}(y_0)+O((y_1-y_{-1})^5).$$

By means of rules for finding the derivative function in the implicit case, it is easily seen that